TECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 66 NUMBER 4

CONTENTS

Some Uses of Linear Spaces in Analysis . . . . F. A. FICKEN 259
Khintchine's Constant . DANIEL SHANKS AND J. W. WRENCH, JR. 276
Note on the Distributive Laws . . . . . TÔRU SAITÔ 280
The Polynomial Correlation Coefficient . . . . . W. D. BATEN AND J. S. FRAME 283
Mathematical Notes. . . . G. N. COPLEY, JACK LEVINE, M. S. KLAMKIN AND D. J. NEWMAN, J. C. C. NITSCHE, A. B. FARNELL 288
Classroom Notes. . . . . DAVID ZEITLIN, ARTHUR PORGES, H. J. ZIMMERBERG 300
Mathematical Education Notes . . W. C. KELLY, JOHN WAGNER 307
Elementary Problems and Solutions 312
Advanced Problems and Solutions 317
Recent Publications—College Textbooks 323
News and Notices 349
The Mathematical Association of America 353
The Forty-second Annual Meeting 353
November Meeting of the Philadelphia Section 355
December Meeting of the Maryland-D.C.-Virginia Section . . . 355
Officers and Committees as of February 1, 1959 357
Calendar of Future Meetings 360

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The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

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THIS IS THE OFFICIAL JOURNAL OF THE

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For non-members the subscription price is $6.00.

Published by the Association at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June-July,
August-September, October, November, December.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.
Second-class postage paid at Menasha, Wisconsin.

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PRINTED IN THE UNITED STATES OF AMERICA

SOME USES OF LINEAR SPACES IN ANALYSIS*

F. A. FICKEN, University of Tennessee

During recent decades much progress in analysis has resulted from the application of algebraic and topological ideas to classes of functions. How does it happen that these ideas, which arise so naturally in Euclidean space of dimension two or three, can be extended so usefully to classes of functions? We shall try to throw some light on this question by describing and illustrating a few of the conceptual relationships involved.

Much of the usefulness of linear spaces arises from very simple algebraic facts. Let D denote any set and let R denote the class of all real-valued functions defined on D, with the understanding that two functions are equal if their values are equal at every point of D. For example, if D is the set of positive integers, then R is the class of all real sequences. If D is the real line, or a half line, or an interval, we have the class of all real-valued functions of a real variable with domain D.

There is an obvious way of adding elements of R. If $u \in R$ and $v \in R$, and the function $w = u + v$ is defined by agreeing that, for every $P \in D$

$$w(P) = (u + v)(P) = u(P) + v(P),$$

where the last expression is the numerical sum of the numbers $u(P)$ and $v(P)$, then clearly $w \in R$. It is evident that R with this operation of addition is an abelian group; the identity is the zero function, which vanishes identically on D.

Elements of R can also be multiplied by real numbers in the same pointwise manner. If $u \in R$ and a is a real number, and the function $v = au$ is defined by agreeing that, for every $P \in D$

$$v(P) = (au)(P) = au(P),$$

where the last expression is the numerical product of the number a and the number $u(P)$, then clearly $v \in R$. One verifies readily that the abelian group just mentioned "admits" the operation of multiplication by real numbers in the sense that the following equalities hold for all elements u and v of R and all real numbers a and b:

$$a(u + v) = au + av, \qquad (a + b)u = au + bu,$$
$$a(bu) = (ab)u, \qquad 0u = 0, \qquad 1u = u.$$

Under these circumstances R is said to be a real linear or vector space. ([1], p. 162; [2], Sec. 2; [3], Sec. 1.4; [4], p. 6; [5], Sec. 1.2. These and following similar indications refer to page or section of the numbered item in the bibliography.) The elements of R (the functions u) are the vectors and the real numbers are the scalars. When reference is made to a space of functions with a common

* An invited address at the Southeastern Sectional Meeting of the Mathematical Association of America, held at the University of Florida, March 14–15, 1958.

domain, it will always be understood that this natural algebraic structure is being used. The functions will always be real-valued; thus we deal exclusively with real linear spaces of functions. It may also be noted that we are thinking in terms of the affine rather than the projective model of a linear space.

We first review Picard's theorem for the differential equation $y'=f(x, y)$ in such a way as to arrive at the notion of a complete metric linear space. Next we suggest considerations, first algebraic and then topological, underlying the idea of a Banach space. Then the use of Banach spaces is illustrated and, finally, some hint is given of the rich properties of Hilbert space.

Our aim is strictly expository, and proofs will be found in the references. It should also be emphasized that we can mention only a few fragments, chosen somewhat capriciously from a vast area of which many parts are still being studied vigorously. Being interested primarily in conceptual relationships, we always use hypotheses strong enough to exclude every kind of irregularity.

Picard's theorem for $y'=f(x, y)$. Let us review briefly the main features of Picard's familiar treatment of the differential equation $y'=f(x, y)$ ([6], Ch. XV; [7], Secs. 6.9–6.11). Let x range over the interval $I: \alpha<x<\beta$. Let $f(x, y)$ have these properties:

1. *$f(x, u)$ is continuous in x and u together and $|f(x, u)| \leqq M$ for $x \in I$ and all u;*
2. *$|f(x, v)-f(x, u)| \leqq K|v-u|$ for $x \in I$ and all u and v (Lipschitz condition).*

Let a and b be given numbers such that $a \in I$. The problem (P) is to find a differentiable function $y(x)$ such that

$$\text{(P)} \qquad y'(x) = f(x, y(x)) \qquad (x \in I) \text{ and } y(a) = b.$$

The first step is to pass by integration to an integral equation satisfied by any solution of (P):

$$\text{(E)} \qquad y(x) = b + \int_a^x f(t, y(t))dt.$$

Then observe that, vice versa, a continuous function satisfying (E) will solve (P). Thus (E) and (P) are equivalent.

One then solves (E) by an iterative process, putting $y_0(x)=b$, and, for $n=0, 1, 2, \cdots$.

$$y_{n+1}(x) = b + \int_a^x f(t, y_n(t))dt.$$

The functions $y_n(x)$ are the partial sums of the infinite series

$$\text{(S)} \qquad y_0(x) + \sum_{n=1}^{\infty} (y_n(x) - y_{n-1}(x)).$$

The terms of this series can easily be estimated. First of all,

$$| y_1(x) - y_0(x) | = \left| \int_a^x f(t, b)dt \right| \leqq M | x - a | .$$

By using the properties of $f(x, u)$, one then obtains by induction the estimate

$$| y_{n+1}(x) - y_n(x) | \leqq MK^n \frac{| x - a |^{n+1}}{(n + 1)!} \qquad (n = 1, 2, \cdots).$$

Being dominated by the series for $MK^{-1}e^{K|\beta-\alpha|}$, the series (S) converges (absolutely and) uniformly to a function $y(x)$ which is continuous on I. One sees readily that $y(x)$ satisfies (E) and therefore also solves (P), and is, moreover, the unique solution of (P).

Let us examine certain features of the foregoing process. We are working, let us first observe, with functions y of the independent variable x that are continuous on I. To abbreviate, let C denote the class of those functions; that is, $y \in C$ if and only if y is continuous on I. If $y \in C$ then the function whose value at x is $b + \int_a^x f(t, y(t))dt$ belongs to C. Let us denote this function by Fy; thus, for $x \in I$,

$$Fy(x) = b + \int_a^x f(t, y(t))dt.$$

We therefore have a correspondence or transformation $F: y \to Fy$, defined on C into C, carrying y onto Fy. The equation (E) asks for a self-corresponding element of C, an element y such that $y = Fy$, a fixed point of F.

The natural algebraic structure of C as a real linear space permits us to carry out formally each step in the iterative process by which (E) is solved; it is clear, indeed, that each y_n exists and belongs to C. The algebraic structure, however, gives us no means of deciding whether the y_n are leading us anywhere at all, let alone toward a solution of (E). A notion of convergence is needed, that is, a topology.

The convergence actually used in our proof was uniform convergence on I. One is tempted to try to introduce an equivalent metric topology. A scalar-valued function of one or more vectors in a linear space is called a functional. Consider now the functional

$$\rho(u, v) = \sup_x | u(x) - v(x) |$$

with u and v in C. This symmetric functional is never negative, vanishes if and only if $u = v$, and obeys the triangle inequality. There is this difficulty, however, that if u is bounded and v is unbounded then $\rho(u, v)$ is not finite. If we wish a set on which ρ is always finite, and wish also to keep the algebraic structure, we must insist on keeping the zero of the linear space and require therefore that

$$\rho(u, 0) = \sup_x | u(x) | < \infty;$$

in other words we must use only that subclass B of C consisting of those functions which are bounded as well as continuous on I. B consists of precisely those elements of C which are at a finite distance ρ from the origin.

Using the same operations of addition and multiplication by a scalar, we easily verify that B is itself a vector space; B is therefore called a subspace of C. On B, moreover, $\rho(u, v)$ is a metric such that $\rho(u_n, u) \to 0$ as $n \to \infty$ if and only if $u_n(x) \to u(x)$ uniformly on I. Thus B is a metric linear space. It is obvious, too, that the iterative process for solving (E) took place entirely in B; $y_0 \in B$, and if $u \in B$ then $Fu \in B$.

One final point must be clarified. The functions y_n turned out to be a Cauchy sequence uniformly on I. We may now express this by saying that y_n is a Cauchy sequence in B; in other words $\rho(y_m, y_n) \to 0$ as m and $n \to \infty$. We must be able to conclude that then there exists a y *in* B such that $\rho(y_n, y) \to 0$ as $n \to \infty$. Here we appeal to a theorem of analysis which says that if a sequence of bounded continuous functions on I is a Cauchy sequence uniformly on I then there exists a bounded continuous function to which the sequence converges uniformly on I. In other words B is a metric linear space which is complete in the sense that every Cauchy sequence of elements of B converges to an element of B.

Summarizing, we may say that the integral equation (E) asks for a fixed point of the transformation $y \to Fy$ defined on the complete metric linear space B into itself, and that the Picard process locates a fixed point by an iterative process.

Algebraic considerations. Having seen that a complete metric linear space arises quite naturally in certain problems in analysis, let us return for a closer look at some algebraic properties of the linear space $R(D)$ of all real-valued functions defined on an arbitrary set D.

If a property is preserved under the formation of linear combinations then the subset of R consisting of all functions with that property will form a subspace of R. It is fortunate indeed that many properties of interest in analysis are in fact preserved under the formation of linear combinations. This is true, for example, of such regularity properties as boundedness, continuity, Hölder continuity (with fixed exponent), smoothness, and integrability. It is true of measurability, and of evenness and oddness, but not of convexity. It is true of the property of vanishing on a given subset of D. It is true of the property of satisfying a linear homogeneous differential or integral equation.

These examples lead us to the notion of a linear transformation. Let X and Y be real linear spaces and let L be a mapping $x \to y = Lx$ on X into Y that is linear: $L(a_1x_1 + a_2x_2) = a_1Lx_1 + a_2Lx_2$. If X and Y are finite-dimensional, and bases are introduced in them, then a matrix L will exist such that in the equation $Lx = y$ the column vector x is multiplied by the matrix L to yield the column vector y.

Let us now fix $y \in Y$ and seek to solve the equation $Lx = y$ for x; in the finite-

dimensional case we are trying to solve a system of linear equations $\sum_{j=1}^{n} l_{\alpha j}x_j = y_\alpha$ $(\alpha=1, \cdots, m)$.

If $y=0$ (homogeneous case) the trivial solution $x=0$ is always available. If $y=0$ implies $x=0$ then L is said to be nonsingular; otherwise L is said to be singular. If L is singular then the set of solutions of $Lx=0$ is a subspace of X called the nullspace of L and denoted by N_L. If $y\neq 0$ then the equation $Lx=y$ is consistent (has a solution) if and only if y is in the range of L; in the finite dimensional case the condition is that y be in the column space of the matrix L, so that L and the "augmented matrix" (with extra column y) have the same rank. In any case we may easily verify that *either* L is nonsingular and for each y in the range of L there is a unique x such that $Lx=y$ *or* L is singular and $Lx=y$ if and only if $x=x_0+n$ where $Lx_0=y$ and $Ln=0$, that is, $n\in N_L$.

In geometric language, the set of all solutions is obtained by translating the subspace N_L by any particular vector x_0 satisfying the equation. For want of a better name, let us for present purposes only call a set arising from translation of a subspace a flat. The result then is that the set of solutions of $Lx=y$ is the flat obtained by translating N_L by any particular solution x_0.

Generally speaking, now, any linear condition can be expressed by a linear equation $Lx=y$ where L is defined on an appropriate domain into an appropriate range. As a first example, let us consider a linear differential operator on an interval I:

$$x \to Lx = a_n(t)x^{(n)}(t) + \cdots + a_1(t)x'(t) + a_0(t)x(t)$$

with continuous coefficients and $a_n(t)\neq 0$ on I. For certain problems an appropriate domain is the space C^n of functions with n continuous derivatives on I, and the range is the space C^0 of functions continuous on I. It is known that the nullspace N_L is n-dimensional; a basis for N_L consists of n linearly independent solutions $x_1(t), \cdots, x_n(t)$ of the equation $Lx=0$, and a parametric representation

$$x_c(t) = c_1x_1(t) + \cdots + c_nx_n(t)$$

of N_L is commonly called the "complementary function." It may also be shown, by variation of parameters, that the range of L is all of C^0. It follows from our general remarks, therefore, that for any y in C^0 the nonhomogeneous equation $Lx=y$ is consistent and its solutions constitute the flat $x(t)=x_p(t)+x_c(t)$, where $Lx_p(t)=y(t)$; $x_p(t)$ is commonly called a "particular integral" ([8], Ch. 3, Sec. 6).

The same general principle underlies the "displacement of nonhomogeneity" in the problem of finding a function subject to several linear conditions, some of which are nonhomogeneous ([9], p. 236). Suppose, for example, that the conditions fall into two sets, which may be written symbolically as

$$Lu = H, \qquad lu = h,$$

where H and h are given. For example, L might be a differential operator, while

the equation $lu=h$ might express initial or boundary conditions. Suppose further that one set of conditions can be met, for example, that $lu_0=h$. Put $u=u_0+v$, getting for v the conditions

$$Lv = H - Lu_0, \qquad lv = 0.$$

Thus v must be sought in the flat determined by translating by u_0 the nullspace of conditions l.

For a final example, let us return to the space $R(D)$ of all real-valued functions defined on a domain D. Let A be a nonvoid proper subset of D, let $y(P)$ be defined only on A, and consider the nonhomogeneous problem of finding those $x\in R(D)$ such that $x(P)=y(P)$ for $P\in A$. This problem amounts to finding all extensions of y from A to D. Since $y\in R(A)$, let us consider a transformation L on $R(D)$ into $R(A)$ defined by $x\to Lx$, where Lx is the element $x(P)$ of $R(A)$, sometimes called the restriction of x to A. In this case the subspace N_L of $R(D)$ consists of all functions defined on D so as to vanish on A, and the solution of the extension problem is the flat consisting of N_L translated by any particular extension of y (for example, one obtained by putting $x=0$ on the complement of A).

These remarks apply to any problem in which a desired function is required, among various constraints, to take on given values at specified points. In the most elementary problem in the calculus of variations, for example, one is interested in all sufficiently regular functions $x(t)$ taking on prescribed values at, say, $t=a$ and $t=b$; from one such function x_0 one obtains all others by translating by x_0 the space of all sufficiently regular functions vanishing at a and at b. Initial and boundary value problems for ordinary and hyperbolic or parabolic partial differential equations, as well as the Dirichlet and Neumann problems for elliptic equations all require functions to take on specified values at specified places. In all these cases, therefore, the desired function must be sought within an appropriate flat.

Another algebraic idea has important consequences ([1], Ch. VII, Sec. 12; [2], Sec. 13; [4], Ch. I, Sec. 10). A real-valued linear function $x^*(x)$ defined on a real linear space X is called a linear functional; thus x^* is a linear transformation on X into the one-dimensional space of scalars. In a finite-dimensional space, in a fixed coordinate system, each linear functional determines a unique linear form, and vice-versa. In the space C^0 of all functions $x(t)$ continuous for $\alpha\leqq t\leqq\beta$, $x^*(x)=\int_\alpha^\beta x(t)dt$ is a linear functional. In the space $R(D)$, for each fixed $P\in D$, $x_P^*(x)=x(P)$, the value of x at P, is a linear functional on $R(D)$; the extension problem just discussed seeks those $x\in R(D)$ such that $P\in A$ implies $x_p^*(x)=y(P)$.

The class of all linear functionals, with addition and multiplication by real numbers again defined pointwise, is a linear space X^* called the dual of X.

If $x\in X$ and $x^*\in X^*$ then x and x^* are said to be normal to each other if and only if $x^*(x)=0$. If Y is a subspace of X then the set of all $x^*\in X^*$ such that

$y \in Y$ implies $x^*(y) = 0$ is easily seen to be a subspace of X^*; this subspace will be denoted by Y^N. Similarly, to any subspace Z of X^* there corresponds the subspace Z^N of X (not of X^{**}) consisting of all $x \in X$ normal to every $x^* \in Z$.

We have seen that the solutions of a system of linear homogeneous algebraic equations $Lx = 0$ constitute the nullspace N_L of the transformation L. If in a given coordinate system the equations require certain linear forms to vanish

$$x_\alpha^*(x) = \sum_{j=1}^{n} l_{\alpha j} x_j = 0 \qquad (\alpha = 1, \cdots, m)$$

and if Z is the subspace of X^* spanned by the x_α^*, then the solutions constitute the space Z^N of X; thus $Z^N = N_L$.

Let us conclude these algebraic observations by mentioning some of the difficulties that arise in the infinite-dimensional case. In the finite-dimensional case, for example, it is always true that

$$U^{NN} = U \quad \text{and} \quad (U \cap V)^N = U^N + V^N.$$

In the infinite-dimensional case, if U and V are subspaces of X^*, then only the inclusions

$$U \subset U^{NN} \qquad U^N + V^N \subset (U \cap V)^N$$

can be established generally ([5], p. 50 (Prop. 10); p. 48 (Prop. 8); p. 60 (9c)).

In the finite-dimensional case, again, X and X^* and X^{**} are isomorphic. It is a consequence of the definition of dimension in the infinite case that if C denotes the cardinal of the real number system, then $\dim X^* = C^{\dim X} > \dim X$, so that isomorphism between X and X^* or X^{**} is out of the question ([4], p. 247).

The symmetry in the relationships between normal subspaces is restored, and dimensional questions are again brought under control, by adjustments in the notion of the dual space required by topological considerations, to which we now turn.

Topological considerations. A topological linear space X is a linear space with a topology in which the operations of addition and multiplication by a scalar are continuous. We cannot discuss the general theory of these spaces, which has been developed extensively in recent years [10]. If the topology in X is induced by a metric $\rho(u, v)$ then one speaks of a metric linear space.

In order to see that the most general metric linear space may be mildly pathological let us ask which linear functionals x^* are continuous in the sense that $\rho(x_n, x) \to 0$ implies $x^*(x_n) \to x^*(x)$. Let M denote the metric linear space of measurable functions $x(t)$ on the unit interval with the metric

$$\rho(x, y) = \int_0^1 \frac{|x(t) - y(t)|}{1 + |x(t) - y(t)|}\, dt.$$

The only continuous linear functional on M is the functional $x^*=0$, *i.e.*, $x^*(x)=0$ for all x ([11], p. 234).

Let us recall that a subset of a linear space is convex if it contains the entire segment joining any pair of its elements. Now it has been shown that nontrivial continuous linear functionals exist in a topological linear space if and only if the origin is contained in an open convex proper subset ([12], p. 18; [13]). In a metric topology, a sufficiently small open sphere with center at the origin will meet this condition provided the sphere is convex; in the space M, therefore, all sufficiently small spheres containing the origin are not convex.

Difficulty arises on an even more elementary level when one tries to connect the metric $\rho(u, v)$ with the notion of length of a vector. It is natural to investigate the effect of defining the length of u to be $\rho(u, 0)$. One would wish the length of $v-u$ to be $\rho(u, v)$, and would wish the ratio of the length of au to the length of u to be $|a|$. The metric in the space M does not have these properties.

It is perhaps not entirely unnatural that in a vector space the notion of length of a vector is the decisive metric feature. A real-valued functional $\|u\|$ is said to be a *norm*, and its value at u is called the length of u if the following conditions are satisfied:

1. $\|0\|=0$, and if $u\neq 0$ then $\|u\|>0$;
2. $\|au\|=|a|\,\|u\|$;
3. $\|u+v\|\leqq\|u\|+\|v\|$.

In the space of bounded continuous functions on a domain D the functional $\|u\|=\sup_{x\in D}|u(x)|$ is a norm; many other examples will occur.

It is easy to see that, if $\|u\|$ is a norm then $\rho(u, v)=\|v-u\|$ is a metric. The resulting metric topology is called the normed topology. Clearly $u_n\to v$ if and only if $\rho(u_n, v)=\|u_n-v\|\to 0$. It has been shown that, in a topological linear space, a norm can be defined yielding an equivalent topology if the origin has a bounded convex symmetric neighborhood ([12], p. 18; [14]).

In order to be able to carry out limiting operations freely we require that our normed linear space be complete; that is, if $\|u_m-u_n\|\to 0$ as $m, n\to\infty$ then there exists an element v of the space such that $\|u_n-v\|\to 0$ as $n\to\infty$. If a normed linear space arises which is not complete, it can always be completed in a unique way by a process very similar to the Cauchy-Cantor process for generating the reals from the rationals ([15], p. 196; [16], p. 94 (Th. 2); [18], p. 30). A complete normed linear space is usually called a Banach space.

Many of the topological difficulties encountered in infinite-dimensional Banach spaces may be traced to the fact that the unit sphere $\|x\|\leqq 1$ is not compact. A locally compact Banach space is finite-dimensional ([11], p. 84).

The advantages of a norm appear clearly in connection with the dual space. The (topological) dual X^* of a *topological* linear space X is now defined to consist of all linear functionals x^* that are *continuous* on X. It was observed above that X^* may be trivial; M^* is. If X is a normed linear space, however, the situation is very satisfying.

Let us observe once and for all that any linear transformation L on one Ba-

nach space into another (and in particular any linear functional x^*) is continuous if and only if the nonnegative real number

$$\|L\| = \sup_{x \neq 0} \frac{\|Lx\|}{\|x\|} \qquad \left(\text{or } \|x^*\| = \sup_{x \neq 0} \frac{|x^*(x)|}{\|x\|}\right)$$

is finite ([16], p. 134 (Th. 2)). Then the transformation (or functional x^*) is said to be bounded and $\|L\|$ (or $\|x^*\|$) is called its norm.

Thus X^* is the space of bounded linear functionals, and it turns out that $\|x^*\|$ does have, as its name implies, the properties of a norm on X^*([16], p. 149). Whether or not X is complete in its norm $\|x\|$, moreover, X^* is complete in the norm $\|x^*\|$ ([16], p. 149 (Th. 1)). Thus the dual X^* of a Banach space X comes with a built-in normed topology with respect to which it is itself a Banach space.

Finally, X^* is definitely nontrivial, as indicated by the fact that if $x_0 \neq 0$ then there exists an $x^* \in X^*$ such that $x^*(x_0) = \|x_0\|$ and $\|x^*\| = 1$ ([16], p. 146 (Th. 5); [12], p. 19 (Th. 2.9.3)).

As an example, let us mention the space L^p $(p > 1)$ of functions $x(t)$ on a finite interval I with the property that $|x(t)|^p$ is integrable over I in the Lebesgue sense, with the norm $\|x\|$, where $\|x\|^p = \int_I |x(t)|^p dt$. The space L^{p^*} dual to L^p is isomorphic and isometric with L^q, where $p+q = pq$. It can be shown, in fact, that for each x^* in L^{p^*} there exists a unique $y \in L^q$ such that, for all $x \in L^p$, $x^*(x) = \int_I y(t)x(t)dt$ ([16], p. 141 (Th. 3); [11], p. 64). The space L^2, to which we shall return later, is self-dual.

The spaces L^p have many pleasant properties. For example, $(L^{p^*})^* = (L^q)^* = L^p$. In order to mention one rather disconcerting property of L^p for $p \neq 2$, let us recall that subspaces X and Y of a linear space Z are said to be complementary if $X \cap Y = 0$ and for each $z \in Z$ there exist $x \in X$ and $y \in Y$ such that $z = x + y$. It is an algebraic fact that every subspace has many complements ([4], pp. 27–8; [5], p. 37 (Prop. 5)). The question here concerns the existence of complementary closed subspaces. It has been shown that if $p \neq 2$ then L^p has a closed subspace S with no closed complement [17]. As a corollary, there is no bounded projection of L^p onto S.

The Picard process led us to a Banach space. After reviewing briefly some basic algebraic and topological ideas we have reached the same point again, with perhaps a clearer idea of the structure of a Banach space. Our object in the remaining sections is to illustrate a very few of the uses of Banach spaces and of a very important special class of Banach spaces, namely, Hilbert spaces.

Banach spaces. Let us mention first a theorem on the iterative solution of equations $x = Tx$, where $x \to Tx$ is a mapping (nonlinear, to avoid triviality) on a Banach space X into itself ([18], p. 58). Suppose that, for some $r > 0$, $\|x\| \leqq r$ implies $\|Tx\| \leqq r$; thus T maps into itself the closed sphere of radius r and center at the origin. Suppose further (Lipschitz condition) that there is a number θ with $0 < \theta < 1$, such that $\|x\| \leqq r$ and $\|y\| \leqq r$ imply $\|Ty - Tx\| \leqq \theta\|y - x\|$. Then

there is a unique x such that $\|x\| \leqq r$ and $x = Tx$. One proves this theorem by first choosing x_0 with $\|x_0\| \leqq r$, then, for $n = 0, 1, 2, \cdots$, defining $x_{n+1} = Tx_n$, then showing that x_n is a Cauchy sequence, and showing finally that $x = \lim_n x_n$ satisfies the equation and is unique. The analogy with Picard's process is clear.

Before applying this theorem let us note that there are many other fixed-point theorems. The one just cited necessarily yields a unique result (with $\|x\| \leqq r$) and therefore can not be applied if there is more than one solution (with $\|x\| \leqq r$).

One specific application of an iterative process of the kind just described has resulted in a treatment of the following problem:

$$\begin{cases} u_{xx} - u_{tt} - 2ru_t - \alpha u = b + \epsilon u^3 & (-\infty < x < \infty, 0 < t) \\ u(x, 0+) = f(x),\ u_t(x, 0+) = g(x) & (-\infty < x < \infty), \end{cases}$$

where f has two continuous derivatives and g has one. This is the initial value problem for the transverse displacement of an infinite string or for the voltage in an infinite transmission line, with applied force $-b$. The nonlinear term ϵu^3 is analogous to the nonlinear term in Duffing's equation. One first "solves" the differential equation as though it were a nonhomogeneous "telegraph equation," and uses the initial conditions, arriving at an equivalent single nonlinear integral equation $u = Tu$, where

$$\begin{aligned} Tu(x, t) &= \tfrac{1}{2}e^{-rt}\int_{-t}^{t} [rf(x+y) + g(x+y)]J_0(\omega\sqrt{t^2 - y^2})dy \\ &+ \tfrac{1}{2}e^{-rt}\frac{\partial}{\partial t}\int_{-t}^{t} f(x+y)J_0(\omega\sqrt{t^2 - y^2})dy \\ &- \tfrac{1}{2}\int_0^t e^{-rs}\int_{-s}^{s} [b(x+y, t-s) + \epsilon u^3(x+y, t-s)]J_0(\omega\sqrt{s^2 - y^2})\,dyds; \end{aligned}$$

here J_0 is the Bessel function of order zero and $\omega^2 = \alpha - r^2$.

Let us keep the present discussion simple by assuming that $r > 0$. Then the Banach space appropriate for an iterative solution of the equation $u = Tu$ is the space CC of functions $w(x, t)$ that are bounded and continuous in x and t together for $-\infty < x < \infty$ and $t \geqq 0$, with $\|w\| = \sup_{x,t} |w(x, t)|$. One arrives in this way at a rather complete discussion of the initial value problem. It is possible to show, moreover, that if the applied force is p-periodic in t, $b(x, t+p) = b(x, t)$, then the initial data f and g can be chosen so that the corresponding solution is also p-periodic in t [19].

This method of dealing with the infinite string has the advantage that the semi-infinite string with fixed endpoint and the finite string with both endpoints fixed can be regarded as special cases. To treat the semi-infinite string with fixed endpoint, at $x = 0$, one goes into the subspace of CC consisting of functions that are odd in x. To treat the finite string with fixed endpoints, at $x = 0$ and $x = \pi$, one goes into the subspace of CC consisting of functions that are odd and

2π-periodic in x. These subspaces are easily seen to be invariant under the integral operators that are used. If begun in the appropriate subspace, the iterative process will therefore take place entirely within the subspace, and, since the subspaces are closed, will deliver a result also lying there.

Another remark concerns the passage from a differential equation with accompanying initial (or boundary) conditions to a single integral equation. It is true that getting the problem into a single package may be advantageous. More to the point, however, is the general experience that many integral operators, including most of the linear ones that seem to arise in practice, are bounded and therefore continuous, while a differential operator, in the nature of things, is necessarily unbounded.

For example, let C^0 denote the space of functions $x(t)$ continuous for $0 \leqq t \leqq 1$, with $\|x\| = \sup_t |x(t)|$, let $K(s, t)$ be continuous for $0 \leqq s, t \leqq 1$, with $k = \sup_{s,t} |K(s, t)|$, and define a linear transformation $x \to Tx$ on C^0 into C^0 by putting

$$Tx(t) = \int_0^1 K(t, s)x(s)ds.$$

Since $|Tx(t)| \leqq k\|x\|$, it follows that $\|T\| \leqq k$ and T is bounded with norm not exceeding k. On the other hand, the transformation defined on the subspace C^1 of C^0, consisting of those functions with derivatives lying in C^0, by $x \to Dx$, where $Dx(t) = dx(t)/dt$, is not bounded, for if $x_n(t) = t^n$, then $\|x_n\| = 1$ and $\|Dx_n\| = \sup_t nt^{n-1} = n$.

Another reason for preferring integral to differential operators is their smoothing effect. Thus the transformation $x \to Jx$, where $Jx(t) = \int_0^t x(u)du$ maps C^0 into the subspace C^1, whose elements are more regular than those of C^0, while $x \to Dx$ is defined (for our purposes) only on the subspace C^1 and yet the range of D covers all of C^0; in fact, for any $x \in C^0$, $Jx \in C^1$ and $DJx = x$.

The notion of the dual transformation of a given linear transformation is illuminating in the general as well as in the finite-dimensional case ([12], Sec. 2.13; [16], Ch. 7, Sec. 13). Let L map a Banach space X linearly and continuously into a Banach space Y, and let R_L denote the range LX of L. Being a subspace of the Banach space Y, R_L is complete, and therefore a Banach space, if and only if R_L is closed ([16], p. 95). Let X^* and Y^* denote the Banach spaces dual respectively to X and Y; thus Y^* is the space of bounded linear functionals y^* defined on Y. Now observe that, with y^* fixed in Y^*, the number $y^*(Lx)$ depends linearly on x; moreover, since

$$|y^*(Lx)| \leqq \|y^*\| \|L\| \|x\|,$$

$y^*(Lx)$ is a bounded linear functional of x and therefore is an element x^* of X^*. This element x^* of X^* depends linearly on y^*. The linear transformation on Y^* into X^* carrying y^* onto x^* is called the dual L^* of L; the equations defining L^* are therefore $x^* = y^*L^*$, where $y^*L^*(x) = y^*(Lx)$.

In the finite-dimensional case, using suitable bases, let us think of the vectors of X and Y as column vectors and of the vectors of X^* and Y^* as row vectors. Then the applicable equations between matrices are $Lx=y$ and $y^*L^*=x^*$, where the matrix L^* is identical with the matrix L ([4], p. 58). If one wishes to treat x^* and y^* also as column vectors then the second equation becomes $x^{*t}=L^t y^{*t}$, where L^t is the transpose of L; this is the basis for the conventional identification of the dual with the transpose.

Recalling now in the general case that the nullspace of L is denoted by N_L, we may display the relationships ([16], Ch. 10, Sec. 3; [5], Sec. 4, No. 9) between the ranges and nullspaces of L and L^* as follows:

$$N_L = (R_{L^*})^N; \qquad N_{L^*} = (R_L)^N;$$

$$R_L \subset (N_{L^*})^N, \text{ and conversely if } R_L \text{ is closed;}$$

$$R_{L^*} \subset (N_L)^N, \text{ and conversely if } R_{L^*} \text{ is closed.}$$

In order to obtain a typical application, let us suppose that R_L is closed (as it always is, of course, in the finite-dimensional case). Then $R_L=(N_{L^*})^N$, and we have arrived in the general case at a consistency condition familiar in the finite-dimensional case for the equation $Lx=y$; if R_L is closed then $y\in R_L$ if and only if y is normal to every solution of the homogeneous dual ("transposed") equation $y^*L^*=0$.

As a final example, let us mention very briefly the notion of the differential of a transformation $x\to Tx$ on an open subset S of X into Y ([12], Sec. 4.3; [18], p. 127). Suppose that, with x fixed in S, and $x+u$ in S, it is possible to express the increment $T(x+u)-Tx$ as

$$T(x+u) - Tx = L(x; u) + \|u\| R(x; u),$$

where $u\to L(x, u)$ and $u\to R(x, u)$ map X into Y in such a way that L is linear in u and $R(x, u)\to 0$ as $u\to 0$. Then L is said to be the differential of T at x.

As a simple instance of this idea, let X be the space C^1 of functions $x(t)$ that are continuous and have continuous first derivatives on the interval $a\leqq t\leqq b$, let S be the flat consisting of those $x\in C^1$ such that $x(a)=\alpha$ and $x(b)=\beta$, and let U be the subspace of C^1 consisting of those x such that $x(a)=0=x(b)$; thus $x\in S$ and $u\in U$ imply $x+u\in S$. Let Y be the system of real numbers, let $F(t, x, x')$ be a sufficiently regular function of three variables, and define T on S into Y by setting

$$x \to Tx = \int_a^b F(t, x(t), x'(t))dt.$$

One sees readily that

$$L(x; u) = \int_a^b u\left[\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x'}\right]dt,$$

where the arguments of F_x and $F_{x'}$ are t, $x(t)$, and $x'(t)$. The coefficient of u is the Euler expression for the integrand F.

The differential has been used [20] to develop implicit function theorems for equations of the form $T(x; y)=0$, where T maps a product space $X \times Y$ into X, so as to obtain local solutions $x(y)$ of the equations. It is essential for this purpose to have a known pair (x^0, y^0) such that $T(x^0; y^0)=0$. If the differential $L(x^0, y^0; u)$ with respect to x is nonsingular, and certain more detailed conditions are met, then for y near y^0 one can find $x(y)$ such that $T(x(y); y)=0$ and $x(y) \to x^0$ as $y \to y^0$. In the finite-dimensional case the matrix of the differential turns out to be the Jacobian matrix.

Hilbert spaces. A very important special class of Banach spaces consists of the Hilbert spaces ([16], [21], [22], [23], [24]). Forgetting topological notions for a moment, let us suppose that we have a linear space X endowed with a scalar product, that is, with a symmetric bilinear functional, ordinarily written (x, y), such that the associated quadratic form is positive: $(x, x)>0$ unless $x=0$. In this situation one can show readily that the functional $\|x\|=\sqrt{(x, x)}$ is a norm on X. With this norm, after completion, if necessary, X is called a Hilbert space.

In the space C^0 of continuous functions $x(t)$ on the interval $0 \leqq t \leqq 1$, for example, one might define

$$(x, y) = \int_0^1 x(t)y(t)dt,$$

thus arriving at the norm $\|x\|$, where $\|x\|^2=\int_0^1 x^2(t)dt$. With this norm, C^0 is not complete; its completion is the Hilbert space L^2 of all functions whose squares are Lebesgue-integrable on the unit interval ([18], pp. 32–3 (Ex. 3.1)).

We digress briefly to observe that not every norm can be generated by a scalar product. If there were to be a scalar product (x, y) inducing a given norm $\|x\|$, then (x, y) could be calculated from the norm by the formula

$$(x, y) = \tfrac{1}{4}[\|x + y\|^2 - \|x - y\|^2].$$

It follows immediately from the properties of the norm that the functional $f(x, y)$ on the right is symmetric, and that $f(x, x)=\|x\|^2$ is positive. When will $f(x, y)$ be bilinear, and therefore a scalar product? We give the earliest of the many conditions in the literature ([25], [26]): It must be true of any vectors x and y that

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

(parallelogram law). One may verify by obvious examples that the space C^0 mentioned above with the norm $\|x\|=\sup_t |x(t)|$, while assuredly a Banach space, does not obey the parallelogram law.

An essential property of a Hilbert space H is that H is isomorphic and isometric with H^*; this commonly expressed by saying that H is self-dual. The

basic theorem ([16], p. 138; [24], p. 31 (Th. 3)) states that for each x^* in H^* there exists in H a unique y, depending on x^*, such that $\|y\| = \|x^*\|$ and, for each x in H, $x^*(x) = (y, x)$. Vice versa, for each fixed $y \in H$, the functional (y, x) is a bounded linear functional of x, and therefore an element x^* of H^*. Let us say that x^* and y are "associated."

If H is finite dimensional then in each coordinate system there exists a symmetric positive matrix (g_{ij}) such that $(y, x) = \sum_{i,j} y^i g_{ij} x^j$, where y^i and x^j are components of y and x, and for each x^* there exist numbers x_j^* such that $x^*(x) = \sum_j x_j^* x^j$. Thus x^* and y will be associated if $x_j^* = \sum_i y^i g_{ij}$, a relationship which may be expressed in the terminology of tensor analysis by saying that x_j^* and y^i are covariant and contravariant components of the same vector ([27], pp. 56–7; [28], p. 39).

Now let L be a bounded linear mapping $x \to y = Lx$ on H into H. The dual mapping L^* is defined on H^* into H^*, by the equation $x^* = y^* L^*$ where $y^* L^*(x) = y^*(Lx)$. By virtue of the 1-1 correspondence between H and H^* just mentioned, L^* induces a bounded linear mapping L' on H into H; L' is called the adjoint of L. This mapping on H into H has the property that, for every x, $(y, Lx) = (L'y, x)$; this equation may be used, in fact, to define L' on H into H. The adjective "adjoint" is sometimes applied in general to what has here been called the dual L^* (on Y^* into X^*) of a linear mapping L on a linear space X into a linear space Y; there may be some merit in reserving the adjective for the mapping L' on a Hilbert space H into H induced by L^* on H^* into H^* ([16], p. 205).

For example, let L denote a linear ordinary differential operator, and L' its adjoint in the sense usual in the theory of differential equations. Without giving details, which are by no means trivial, let us note that, with the scalar product

$$(x, y) = \int_a^b x(t) y(t) dt,$$

the equation $(x, Ly) = (L'x, y)$ amounts to Green's formula

$$\int_a^b (xLy - yL'x) dt = 0,$$

which is valid for functions satisfying appropriate boundary and other conditions ([6], p. 255; [8], p. 86).

Another consequence of the presence of a scalar product is the notion of orthogonality: x and y are said to be orthogonal if $(x, y) = 0$. It is clear that if the functional x^* and the vector y are associated, so that, for all x, $x^*(x) = (y, x)$, then x^* and x are normal if and only if y and x are orthogonal. Now if Z is a subspace of X then the class of all vectors orthogonal to every vector of Z is a closed subspace $Z^\perp$ of X, and it is evident that the class of functionals associated with vectors of $Z^\perp$ is the subspace of X^* previously called Z^N. In this sense the notion of normality reduces in a Hilbert space to that of orthogonality.

It is clear that if Z is a closed subspace of H then Z and $Z^\perp$ are complementary closed subspaces of H, so that the difficulty noticed earlier in L^p if $p \neq 2$ does not arise for $p=2$. The presence of the distinguished complement $Z^\perp$ is helpful in many investigations.

A Euclidean space of finite dimension d is a Hilbert space, and an essential part of its furniture is a numerous family (if $d \geqq 2$) of orthonormal systems. In any Hilbert space H, one says that a class V of vectors is an orthonormal system if $u \in V$ and $v \in V$ imply

$$(u, v) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v. \end{cases}$$

One says that H is separable if H has a countable subset that is dense in H. It is not hard to show that H is separable if and only if H has a maximal orthonormal system that is countable ([16], p. 95 (Sec. 3)).

Let H be separable, and let v_n $(n=1, 2, \cdots)$ be a maximal orthonormal sequence ([16], p. 113 (Sec. 11); [21], pp. 7–16; [24], Sec. 14). Then each $x \in H$ has a Fourier expansion

$$x = \sum_{n=1}^{\infty} (x, v_n) v_n,$$

where the equation is understood to mean that

$$\left\| x - \sum_{n=1}^{N} (x, v_n) v_n \right\| \to 0$$

as $N \to \infty$. Moreover,

$$(x, y) = \sum_{n=1}^{\infty} (x, v_n)(y, v_n);$$

in particular, $\|x\|^2 = \sum_{n=1}^{\infty} (x, v_n)^2$.

Now the space l_2 of real sequences $s = (s_1, s_2, \cdots, s_n, \cdots)$ such that $\|s\|^2 = \sum_{n=1}^{\infty} s_n^2 < \infty$ is easily seen to be a separable Hilbert space; the scalar product is $(s, t) = \sum_{n=1}^{\infty} s_n t_n$. If H is any separable Hilbert space, with maximal orthonormal sequence v_n, then the mapping $x \to s$ on H into l_2 given by $x \to s = ((x, v_n))$ is easily seen to be 1-1, isomorphic, and also isometric since $\|x\|^2 = \|s\|^2$. Thus every separable Hilbert space is in this sense equivalent to l_2, which is therefore the prototype of the infinite-dimensional separable Hilbert space. In fact, l_2 was the first Hilbert space to be investigated.

In the space L^2 of functions $x(t)$ on the interval $-\pi \leqq t \leqq \pi$ with

$$\|x\|^2 = \int_{-\pi}^{\pi} x^2(t)\,dt < \infty,$$

the integral being taken in the Lebesgue sense, the system of functions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}}\cos nt, \quad \frac{1}{\sqrt{\pi}}\sin nt \qquad (n = 1, 2, \cdots)$$

is a maximal orthonormal system. The Fourier expansion with respect to this system is the familiar Fourier series. Systems of properly normalized orthogonal polynomials on other intervals yield similar examples ([29], [30]). The normalized eigenfunctions of any self-adjoint linear boundary-value problem for an ordinary differential equation on a finite interval constitute a maximal orthonormal set in L^2 on that interval ([8], Ch. 7).

Let us mention an example ([31], Th. 50, 52) in the space L^2 on the interval $0<t<\infty$ with the norm $\|x\|^2=\int_0^\infty x^2(t)dt$. For $x\in L^2$ define the Fourier-Plancherel sine transform as follows:

$$F_s x(t) = \text{l.i.m.}_{a\to\infty} \sqrt{\frac{2}{\pi}} \int_0^a x(u) \sin tu\, du,$$

where the limit is taken in the L^2 norm. It turns out that the mapping $x\to F_s x$ is an isomorphic isometric mapping of L^2 onto itself. Moreover, F_s is involutory: If $y=F_s x$ then $x=F_s y$.

Let us bring these remarks to a close by mentioning very briefly the problem of "diagonalizing" a symmetric (*i.e.*, self-adjoint) transformation L on a Hilbert space H into H ([1], Ch. IX, Sec. 10; [2], Sec. 79; [22], Sec. 92). If H is finite-dimensional then there exist n real numbers $\lambda_1 \leqq \cdots \leqq \lambda_n$ and n mutually orthogonal subspaces $H_1, \cdots, H_n$ such that if $x\in H$ then $x=x_1+\cdots+x_n$ with unique $x_i\in H_i$, and $Lx_i=\lambda_i x_i$ $(i=1, \cdots, n)$. In a coordinate system adapted to the subspaces H_i, the matrix of L has the numbers λ_i down the diagonal and zeros elsewhere, and this representation of L is called its spectral representation. In this coordinate system, $(Lx, x)=\sum_i \lambda_i(x_i, x_i)=\sum_i \lambda_i\|x_i\|^2$.

In order to arrive at a formulation that can be extended readily to the infinite-dimensional case, we let G_t denote the direct sum of those H_i having $\lambda_i<t$, let E_t denote the transformation projecting H onto G_t, and define $\alpha=\inf(Lx, x)$ and $\beta=\sup(Lx, x)$, both taken for $\|x\|=1$. Then for each $x\in H$ the function $\sigma_x(t)=(E_t x, x)$ has these properties:

1. $\sigma_x(t)=0$ for $t\leqq\alpha$, and $\sigma_x(t)=(x, x)$ for $t>\beta$;
2. $\sigma_x(t)$ is nondecreasing and continuous from the left;
3. $\sigma_x(t)$ is constant for $\lambda_i<t<\lambda_{i+1}$ and $\sigma_x(\lambda_i+)-\sigma_x(\lambda_i)=\|x_i\|^2$.

Using the Stieltjes integral, we may then write

$$(*) \qquad (Lx, x) = \int_\alpha^\beta t\, d\sigma_x(t) = \int_\alpha^\beta t\, d(E_t x, x),$$

a relation sometimes expressed by writing $L=\int_\alpha^\beta t\, dE_t$.

One can show ([22], Sec. 107) for any bounded self-adjoint (symmetric) L on any H into H that there exists a one-parameter family of projections E_t such that the function $\sigma_x(t)=(E_t x, x)$ has the first and second of the properties listed

above, and has the third property in an appropriately modified sense. One can still show, moreover, that (*) holds.

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KHINTCHINE'S CONSTANT

DANIEL SHANKS AND J. W. WRENCH, JR., David Taylor Model Basin

Representation by series. The absolute constant of Khintchine [1], [2], [3] may be defined by the slowly convergent infinite product

$$K = \prod_{n=1}^{\infty}\left(1 + \frac{1}{n(n+2)}\right)^{\ln n/\ln 2}, \tag{1}$$

and its logarithm is, therefore, given by the infinite series

$$\ln 2 \ln K = \sum_{n=2}^{\infty} \ln n \ln \frac{(n+1)^2}{n(n+2)}. \tag{2}$$

In view of the equation

$$2 = \prod_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}, \tag{3}$$

we may substitute in the right member of equation (2) first

$$\ln 2 \ln \frac{3}{2} = \sum_{n=2}^{\infty} \ln 2 \ln \frac{(n+1)^2}{n(n+2)},$$

then

$$\ln \frac{3}{2} \ln \frac{4}{3} = \sum_{n=3}^{\infty} \ln \frac{3}{2} \ln \frac{(n+1)^2}{n(n+2)},$$

and finally, by induction, we obtain [4]

$$\ln 2 \ln K = \sum_{n=2}^{\infty} \ln\left(\frac{n}{n-1}\right) \ln\left(\frac{n+1}{n}\right), \tag{4}$$

which can be written alternatively

$$\ln 2 \ln K = -\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n}\right) \ln\left(1 + \frac{1}{n}\right). \tag{5}$$

Now the Cauchy product of the Maclaurin series for $\ln(1-x)$ and $\ln(1+x)$ is readily found to be

$$-\ln(1-x)\ln(1+x) = x^2 + \left(1 - \frac{1}{2} + \frac{1}{3}\right)\frac{x^4}{2} + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right)\frac{x^6}{3} + \cdots, \tag{6}$$

and consequently equation (5) may be written

$$\ln 2 \ln K = \sum_{n=2}^{\infty} n^{-2} + \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3}\right)\sum_{n=2}^{\infty} n^{-4} + \frac{1}{3}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right)\sum_{n=2}^{\infty} n^{-6} + \cdots. \tag{7}$$

This transformation permits the computation of $\ln 2 \ln K$ by use of the appropriate values of the accurately known Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Furthermore, nearly geometric convergence has been attained in equation (7), with the ratio of successive terms tending to a limit of 1/4.

More rapid convergence is obtained by a second application of equation (3), this time in the form

$$\ln 2 = \sum_{n=2}^{\infty} n^{-2} + \frac{1}{2}\sum_{n=2}^{\infty} n^{-4} + \frac{1}{3}\sum_{n=2}^{\infty} n^{-6} + \cdots. \tag{8}$$

Accordingly, the right member of equation (7) can be rearranged to read

$$\ln 2 \ln K = \ln 2 - \frac{1}{2\cdot 3}\left(\ln 2 - \sum_{n=2}^{\infty} n^{-2}\right) - \frac{1}{4\cdot 5}\left(\ln 2 - \sum_{n=2}^{\infty} n^{-2} - \frac{1}{2}\sum_{n=2}^{\infty} n^{-4}\right) - \frac{1}{6\cdot 7}\left(\ln 2 - \sum_{n=2}^{\infty} n^{-2} - \frac{1}{2}\sum_{n=2}^{\infty} n^{-4} - \frac{1}{3}\sum_{n=2}^{\infty} n^{-6}\right) - \cdots. \tag{9}$$

Still more rapid convergence is obtained by deferring the substitution of (6) until after several terms of the series in (5) have been summed. Thus, we can write

$$\ln 2 \ln K = -\sum_{n=2}^{N} \ln\left(1 - \frac{1}{n}\right)\ln\left(1 + \frac{1}{n}\right) + \sum_{n=N+1}^{\infty} n^{-2} + \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3}\right)\sum_{n=N+1}^{\infty} n^{-4} + \cdots. \tag{7a}$$

The generalization of (8), namely

$$\ln\left(1 + \frac{1}{N}\right) = \sum_{n=N+1}^{\infty} n^{-2} + \frac{1}{2}\sum_{n=N+1}^{\infty} n^{-4} + \frac{1}{3}\sum_{n=N+1}^{\infty} n^{-6} + \cdots, \tag{8a}$$

permits us to deduce from (7a) the following rapidly convergent series

$$\ln 2 \ln K = -\sum_{n=2}^{N} \ln\left(1 - \frac{1}{n}\right)\ln\left(1 + \frac{1}{n}\right) + \ln\left(1 + \frac{1}{N}\right) - \frac{1}{2\cdot 3}\left(\ln\left(1 + \frac{1}{N}\right) - \sum_{n=N+1}^{\infty} n^{-2}\right) - \cdots. \tag{9a}$$

Representation by integrals. It is of interest to sum these series in closed form, and this can be accomplished in terms of a definite integral. For, if the coefficients of the terms of the series in (7) are written as follows

$$\begin{aligned}\ln 2 \ln K = &\left(\ln 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots\right)\sum_{n=2}^{\infty} n^{-2} \\ &+ \frac{1}{2}\left(\ln 2 + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots\right)\sum_{n=2}^{\infty} n^{-4} \\ &+ \frac{1}{3}\left(\ln 2 + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \cdots\right)\sum_{n=2}^{\infty} n^{-6} + \cdots,\end{aligned} \tag{10}$$

then, in view of equation (8) and of the relation

$$\frac{1}{2n} - \frac{1}{2n+1} + \frac{1}{2n+2} - \cdots = \int_0^1 \frac{x^{2n-1}}{1+x}\,dx, \tag{11}$$

equation (10) can be written

$$\ln 2 \ln K = (\ln 2)^2 + \int_0^1 \frac{dx}{x(1+x)}\left[\sum_{n=2}^{\infty}\frac{x^2}{n^2} + \frac{1}{2}\sum_{n=2}^{\infty}\frac{x^4}{n^4} + \cdots\right]$$

or

$$\ln 2 \ln\left(\frac{K}{2}\right) = \int_0^1 \frac{dx}{x(1+x)}\left[-\ln\left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{9}\right)\left(1-\frac{x^2}{16}\right)\cdots\right]. \tag{12}$$

The infinite product in the right member of this equation is well known, with relation to the sine function, and we can accordingly write

$$\ln 2 \ln\left(\frac{K}{2}\right) = \int_0^1 \frac{dx}{x(1+x)} \ln\left[\frac{\pi x(1-x^2)}{\sin \pi x}\right]. \tag{13}$$

An alternative form of this result, involving the gamma function, is

$$\ln 2 \ln\left(\frac{K}{2}\right) = \int_0^1 \frac{dx}{x(1+x)} \ln\left[\Gamma(2+x)\Gamma(2-x)\right]. \tag{14}$$

Numerical results. As an illustration of the computational effectiveness of the preceding series, equation (9a) with $N=2$ was employed to evaluate $\ln 2 \ln K$ to more than 65 decimal places. The numerical series may be written

$$\begin{aligned}\ln 2 \ln K = \ln\frac{3}{2} + \ln 2 \ln\frac{3}{2} - &\left\{\frac{1}{2\cdot 3}\sum_{k=2}^{\infty}\frac{S_{2k}''}{k}\right. \\ &\left. + \frac{1}{4\cdot 5}\sum_{k=3}^{\infty}\frac{S_{2k}''}{k} + \frac{1}{6\cdot 7}\sum_{k=4}^{\infty}\frac{S_{2k}''}{k} + \cdots\right\},\end{aligned} \tag{15}$$

where, for simplicity in writing, S_{2k}'' has been used to represent $\sum_{n=3}^{\infty} n^{-2k}$.

Elaborate tables by R. Liénard [5] contain values of $\sum_{n=2}^{\infty} n^{-2k}$ that are correct to 50 decimal places. These data were checked and then extended to at least 65 decimal places, corresponding to $k=1$ (1) 68, by one of the present writers.

Omnibus checks on successive stages of the calculation consisted of using the relations

$$\sum_{k=1}^{\infty} S''_{2k} = \frac{5}{12}, \tag{16}$$

$$\sum_{k=1}^{\infty} \frac{S''_{2k}}{k} = \ln \frac{3}{2}, \tag{17}$$

$$\sum_{r=2}^{\infty} \sum_{k=r}^{\infty} \frac{S''_{2k}}{k} = \frac{5}{12} - \ln \frac{3}{2}. \tag{18}$$

Rounded 65-place approximations to S''_{2k}, when substituted in these checking relations, yielded discrepancies of 3.3×10^{-65}, 0.4×10^{-65}, and 31×10^{-65}, respectively. Subsequent introduction of two additional guard figures in the calculations led to the following approximation, rounded to 66 decimal places.

$\ln 2 \ln K$

$= 0.68472\ 47885\ 63157\ 12329\ 91461\ 48755\ 77762\ 04606\ 75416\ 33744\ 88366\ 06289\ 86781\ 6\cdots.$

Corresponding to this result, the following approximations to Khintchine's constant and its natural logarithm were calculated:

$\ln K = 0.98784\ 90568\ 33810\ 78966\ 92547\ 27147\ 07295\ 43261\ 99254\ 96088\ 67354\ 27755\ 30068\ 7\cdots,$

$K = 2.68545\ 20010\ 65306\ 44530\ 97148\ 35481\ 79569\ 38203\ 82293\ 99446\ 29530\ 51152\ 34556\cdots.$

References

1. A. Khintchine, Metrische Kettenbruchprobleme, Compositio Math., vol. 1, 1934, p. 376.
2. D. H. Lehmer, Note on an absolute constant of Khintchine, this MONTHLY, vol. 46, 1939, pp. 148–152.
3. D. Shanks, Note MTE 164, Math. Tables Aids Comput., vol. 4, 1950, p. 28. (The second formula herein has a typographical error, and should read $s=\ln 2+\sum_{r=3}^{\infty}\cdots$. In an informal communication, D. H. Lehmer sent one of the authors a value of Khintchine's constant calculated correctly to 17 decimal places from this formula.)
4. D. Shanks, Non-linear transformations of divergent and slowly convergent sequences, J. Math. Phys., vol. 34, 1955, p. 40.
5. R. Liénard, Tables Fondamentales à 50 décimales des Sommes S_n, u_n, $\sum_n$, Centre de Documentation Universitaire, Paris, 1948.

NOTE ON THE DISTRIBUTIVE LAWS

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1. A ring is usually defined to be a system R with two compositions, called addition and multiplication, such that

I. R is an abelian group with respect to addition;
II. R is a semigroup with respect to multiplication;
III. R satisfies the distributive laws:

$$x(y+z) = xy + xz, \qquad (y+z)x = yx + zx, \qquad \text{for all } x, y, z \in R.$$

In this note, we discuss first a system S arising from the above definition of a ring by replacing III by the weaker distributive laws:

IV. In S there exist fixed elements e_1 and e_2 such that

$$x(y+z) = xy + xz - e_1, \qquad (y+z)x = yx + zx - e_2, \qquad \text{for all } x, y, z \in S.$$

Such a system is called a w-ring. In this section, we give several lemmas which hold in a w-ring S.

LEMMA 1. $e_1 = 00 = e_2$.

Proof. Setting $x=y=z=0$ in the first equation of IV, we have $00 = 00+00-e_1$. Thus we have $e_1 = 00$. That $e_2 = 00$ can be proved dually.

This common element $e_1 = e_2$ is called the defining element of the w-ring S and in this section we denote it by e.

LEMMA 2. $x0 = e = 0x$ *for all* $x \in S$.

Proof. We have $x0 = x(0+0) = x0+x0-e$. Therefore $x0 = e$ and dually we have $0x = e$.

LEMMA 3. $xe = e = ex$ *for all* $x \in S$.

Proof. We have $xe = x(00) = (x0)0 = e$ and dually we have $ex = e$.

LEMMA 4. $x(y-z) = xy - xz + e$, $(y-z)x = yx - zx + e$ *for all* $x, y, z \in S$.

Proof. We have $xy = x((y-z)+z) = x(y-z) + xz - e$ and so we have $x(y-z) = xy - xz + e$. The second equality can be proved dually.

LEMMA 5. *The set* $\{x \mid xy = yx = e \text{ for all } y \in S\}$ *is a subgroup of the additive group of* S *and it contains the element* e.

Proof. In order to prove the first assertion, it suffices to show that if $py = yp = e$ and $qy = yq = e$ for all $y \in S$, then $(p-q)y = y(p-q) = e$ for all $y \in S$. Indeed, we have $(p-q)y = py - qy + e = e - e + e = e$ and, dually, we have $y(p-q) = e$. The second assertion of the lemma follows from Lemma 3.

In a w-ring S the order of the defining element e with respect to the additive group of S is called the order of S. Evidently, a w-ring S is a ring if and only if $e=0$ and hence if and only if the order of S is 1.

2. In this section we discuss the existence of w-rings. Let n be any positive integer or ∞. Taking an abstract element e we construct the cyclic additive group G_n generated by e and with order n. We define multiplication in G_n as follows:

$$xy = e \qquad \text{for all } x, y \in G_n.$$

Then we have the following

THEOREM 1. *The system G_n constructed above is a w-ring with order n and defining element e. Conversely, any w-ring S of order n has a subsystem which is isomorphic to the system G_n.*

Proof. By definition, G_n is an abelian group with respect to addition. For all $x, y, z \in G_n$ we have $(xy)z=e=x(yz)$ so that G_n is a semigroup with respect to multiplication. Also, for all $x, y, z \in G_n$ we have $x(y+z)=e$, $xy+xz-e=e+e-e=e$, and hence $x(y+z)=xy+xz-e$. Dually, we have $(y+z)x=yx+zx-e$. Thus G_n is a w-ring with defining element e. Moreover, by definition, G_n has order n.

Next we consider an arbitrary w-ring S of order n. By Lemma 5 the set $\{x \mid xy=yx=e \text{ for all } y \in S\}$ contains the cyclic additive subgroup G_n' generated by the defining element e of S. It is clear that, for all $x, y \in G_n'$, we have $xy=e$. Therefore G_n' is closed with respect to addition and multiplication and so is subsystem of S. It is easy to see that G_n' is isomorphic to G_n.

3. Let us consider a w-ring S with defining element e. We now define a new composition $\circ$ as follows:

$$x \circ y = xy - e. \tag{1}$$

Then S becomes a ring with respect to the original addition and the new multiplication $\circ$. In fact, we have $(x \circ y) \circ z=(xy-e) \circ z=(xy-e)z-e$ and so, by Lemmas 4 and 3, we have $(x \circ y) \circ z=xyz-ez+e-e=xyz-e$. On the other hand, $x \circ (y \circ z)=x \circ (yz-e)=x(yz-e)-e=xyz-xe+e-e=xyz-e$. Hence we have $(x \circ y) \circ z=x \circ (y \circ z)$ and thus S is a semigroup with respect to the composition $\circ$. Also we have $x \circ (y+z)=x(y+z)-e=xy+xz-e-e=(xy-e)+(xz-e)=x \circ y+x \circ z$ and dually we have $(y+z) \circ x=y \circ x+z \circ x$. Hence S is a ring with respect to addition and the multiplication $\circ$. Moreover, in this ring we have $e \circ x=x \circ e=0$ for all $x \in S$. In fact, $e \circ x=ex-e=e-e=0$ and dually $x \circ e=0$.

Conversely, let us consider a ring S with multiplication denoted by $\circ$. In S we take an element e such that $e \circ x=x \circ e=0$ for all $x \in S$ and we define a new product as follows:

$$xy = x \circ y + e. \tag{2}$$

Then S becomes a w-ring with respect to the original addition and the new multiplication and e is the defining element of this w-ring. In fact, $(xy)z=(x\circ y+e)z=(x\circ y+e)\circ z+e=x\circ y\circ z+e\circ z+e=x\circ y\circ z+e$. On the other hand, $x(yz)=x(y\circ z+e)=x\circ(y\circ z+e)+e=x\circ y\circ z+x\circ e+e=x\circ y\circ z+e$. Therefore we have $(xy)z=x(yz)$, that is, S is a semigroup with respect to the new multiplication. Also we have $x(y+z)=x\circ(y+z)+e=x\circ y+x\circ z+e=(x\circ y+e)+(x\circ z+e)-e=xy+xz-e$ and dually $(y+z)x=yx+zx-e$.

Summarizing these results, we have the

THEOREM 2. *Let S be a w-ring with defining element e. If we define a new product in S by* (1), *then S is a ring with respect to the original addition and the multiplication $\circ$ and in this ring, $e\circ x=x\circ e=0$ for all $x\in S$. Conversely, let S be a ring with multiplication denoted by $\circ$. If we define a new product in S by* (2), *then S is a w-ring with respect to the original addition and the new multiplication and e is the defining element of this w-ring.*

4. In this section we consider a system T arising from the definition (Sec. 1) of a ring by replacing III by the composite distributive law:

V. $(u+v)(x+y)=ux+uy+vx+vy$ for all $u, v, x, y\in T$.

Such a system is called a c-ring. We first give several lemmas which hold in a c-ring T. In this section we denote 00 by e.

LEMMA 6. $e+e+e=0$.

Proof. Setting $u=v=x=y=0$ in V, we have $e=(0+0)(0+0)=e+e+e+e$ and hence $e+e+e=0$.

LEMMA 7. $x0=0x=e$ *for all* $x\in T$.

Proof. We have $x0=(x+0)(0+0)=x0+x0+e+e$. Cancelling $x0$, we have $x0+e+e=0=e+e+e$. Cancelling $e+e$, we obtain $x0=e$. Dually we have $0x=e$.

LEMMA 8. $x(y+z)=xy+xz-e$, $(y+z)x=yx+zx-e$ *for all* $x, y, z\in T$.

Proof. We have $x(y+z)=(x+0)(y+z)=xy+xz+0y+0z=xy+xz+e+e=xy+xz-e$. Dually, we have $(y+z)x=yx+zx-e$.

COROLLARY. *If a c-ring has an element a such that $a0=0$, then T is a ring.*

In fact, in this case $e=0$ by Lemma 7 and then T is a ring by Lemma 8.

We now obtain the following

THEOREM 3. *A c-ring is a w-ring of order* 3 *or* 1 *according as* $00\neq 0$ *or* $00=0$ *and* 00 *is the defining element of the w-ring. Conversely, a w-ring of order* 3 *or* 1 *is a c-ring.*

Proof. The first assertion follows from Lemmas 6 and 8. Now let S be a w-ring of order 3 or 1. Then $(u+v)(x+y)=(u+v)x+(u+v)y-e=(ux+vx-e)$

$+(uy+vy-e)-e=ux+vx+uy+vy-(e+e+e)=ux+vx+uy+vy$. Thus S is a c-ring.

COROLLARY. *G_3 in Section 2 is a c-ring which is not a ring.*

Appendix. John L. Kelley, *General Topology*, p. 18, gives a definition of a ring as a system which we call a c-ring. The corollaries to Theorem 3 and Lemma 8 above therefore provide a solution to Problem 4784, this MONTHLY, vol. 65, 1958, p. 289. Another solution appears on page 318 of this issue.

THE POLYNOMIAL CORRELATION COEFFICIENT

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Let it be assumed that the variates y and x are related in such a way that the regression of y on x is a polynomial curve. The x-variates are given and are hence fixed. Let the two variates be given in standard units with respect to the sample. This means that their arithmetic means and their sums of squares, respectively, are equal to

$$\bar{y} = \bar{x} = 0 \quad \text{and} \quad \sum x^2 = \sum y^2 = n,$$

where n is the number of pairs of values. Let it also be assumed that the x-variates have their first $2N$ moments equal to those for a normal variable, which means that $\alpha_{2i+1}=0$, $\alpha_{2i}=(2i)!2^i(i!)$, for the fixed interval in which x is defined.

Let the equation of the predicting polynomial at the point x_j be

$$f(x_j) = \sum_{i=0}^{N} a_i x_j^i.$$

For $N=3$, $y=f(x)=a_0+a_1x+a_2x^2+a_3x^3$.

The square of the correlation coefficient between y and $f(x)$ is equal to

$$\mathbf{r}^2 = 1 - \sum (f-y)^2/n$$

for large values of n. By maximizing $\mathbf{r}^2$, four equations arise from which the constants are found to be

$$a_0 = -\gamma/\sqrt{2}, \qquad a_1 = (5r - \sqrt{15}\delta)/2,$$
$$a_2 = \gamma/\sqrt{2}, \qquad a_3 = (\sqrt{15}\delta - 3r)/6,$$

where r is the linear correlation coefficient between observed values of y and x, γ is the correlation coefficient between observed values of y and x^2, and δ is the correlation coefficient between observed values of y and x^3. (Note that the moments, $\sum xy$, $\sum x^2y$, and $\sum x^3y$, have respectively the values nr, $n\gamma\sqrt{2}$, and $n\delta\sqrt{15}$).

The average of the first derivatives of the estimating curve at the given values of x is equal to

$$\bar{f}'(x) = a_1 + 3a_3 = (5r - \sqrt{15}\delta)/2 + (\sqrt{15}\delta - 3r)/2 = r,$$

which indicates that the linear correlation coefficient, r, between y and x is the average of the first derivatives of the predicting curve at the given values of the independent variates. The number of values of the derivative at each x-value depends upon the number of y-values. The averages of the second and third derivatives of the predicting curve at the given values of x are, respectively, $\bar{f}'' = 2a_2$, $\bar{f}''' = 6a_3$.

On substituting the values of a_1, a_2, and a_3 in the expression for $\mathbf{r}^2$ one obtains*

$$\mathbf{r}^2 = (\bar{f}')^2/1! + (\bar{f}'')^2/2! + (\bar{f}''')^2/3! = \sum (\bar{f}^{(i)})^2/i!,$$

which expresses the square of the nonlinear correlation coefficient between observed values of y and x as a function of the averages of the derivatives of the estimating curve at the given values of x.

By use of characteristic functions of bivariate frequency distributions Wicksell† developed an expression for the regression of the mean which is an infinite series involving cumulants and derivatives of the distribution function of the independent variable. The expression for $\mathbf{r}^2$ here discussed also involves moments and averages of the derivatives of $f(x)$.

An example. The following table lists, in standard units, data pertaining to annual volume in cubic feet of Eastern White Pines in relation to their ages. The standard units were calculated before the observations were arranged in this table. The ages are approximately normally distributed. The broken line connects the means of the volume increment arrays as recorded in this table.

It is assumed that the annual increment, y, is a polynomial function of the age x, *i.e.*, $y = a_0 + a_1x + a_2x^2$. The quantities $\sum x^2$ and $\sum y^2$ are not quite equal to 161, the number of pairs, owing to the rounding-off of decimals in the standard units. By the method of least squares the best fitting parabola to the data, before they were arranged in the table, is

$$y = 0.399321 + 0.460474x - 0.398911x^2.$$

By an analysis of variance this parabola was shown to be a significantly better fit to the data than a straight line. A third-degree polynomial was found to be not a significantly better fit than the parabola.

The nonlinear correlation coefficient between y and x, as found from the standard error of estimate, $\sigma_e = \sigma_y\sqrt{(1 - r_{xy}^2)}$, is $\mathbf{r}_{yx} = 0.699$. The nonlinear correlation coefficient as found by the derivatives, or by formula (10) below, is

* For $N = 3$, $\mathbf{r}^2 = (5/2)r^2 - \sqrt{15}\, r\delta + (5/2)\delta^2 + \gamma^2$.

† S. D. Wicksell, Analytical theory of regression, Medd. Lunds Astr. Obs., Series 2, No. 69, 1934. M. G. Kendall, The Advanced Theory of Statistics, London, 1946, Vol. II, pp. 142–145.

$r_{yz}=0.728$. The two values are approximately the same, showing that the non-linear correlation coefficient can be found by a function of the derivatives of the polynomial at the given values of the independent variable.

ANNUAL VOLUME INCREMENT IN CUBIC FEET IN RELATION TO THE AGE OF EASTERN WHITE PINES (STANDARD UNITS)

ANNUAL VOLUME INCREMENT—y \ AGES (MIDPOINTS)	−2.46	−1.97	−1.48	−.98	−.49	0	.49	.98	1.48	1.97	2.46
1.75– 2.25							2	1			
1.25– 1.75					2	4	5	3			
.75– 1.25					5	4	5	2	2		
.25– .75				5	5	9	6	6	1	1	
− .25– .25			1	4	5	7	3	1	6	2	
− .75– −.25			3	3	8	7	5	3	1	2	2
−1.25– −.75			2	4	2		2	2	1		
−1.75–−1.25		2	2	3	1			1			
−2.25–−1.75		1	3								
−2.75–−2.25		1									
−3.25–−2.75	1	1									
−3.75–−3.25	1										
	2	5	11	19	28	31	28	19	11	5	2
Average	−3.25	−2.00	−1.14	−.39	.11	.35	.54	.42	.09	−.10	−.50

The general case. Let the equation of the predicting curve, about which the observed values of y fall be

$$f(x) = \sum_{i=0}^{N} a_i x^i. \tag{1}$$

Let the fixed x values be chosen so that

$$\sum_{j=1}^{n} x_j^i/n = C_i, \tag{2}$$

where $C_{2i+1}=0$ and $C_{2i}=(2i)!/2^i(i!)$. The square of the correlation coefficient between y and $f(x)$ is

$$r^2 = 1 - \sum (f-y)^2/n,$$

since $\sum y^2=n$.

To maximize r^2 with respect to the a_i that appear in f, we equate to zero the

partial derivatives with respect to each a_i. Hence, if we set $f_j = f(x_j)$ we have

$$0 = \sum_{j=1}^{n} [2(f_j - y_j)(\partial f_j/\partial a_i)] = 2\sum_{j=1}^{n}(f_j - y_j)x_j^i$$

or $\sum_j x_j^i f_j = \sum_j x_j^i y_j$.

From the above $r^2 = \sum fy/n$, since $\sum y^2 = n$, $\sum(f^2) = \sum fy$ and $f_j = y|_{x=x_j}$. Hence

$$\begin{aligned} r^2 &= \sum_{i=0}^{N} a_i \sum_{j=1}^{n} [x_j^i f_j]/n \\ &= \sum_{i=0}^{N} a_i \sum_{j=1}^{n}\Big[\sum_{k=0}^{N} a_k x_j^{i+k}\Big]\Big/ n = \sum_{i,k}^{N} a_i a_k C_{i+k}. \end{aligned} \tag{4}$$

Let the average of the kth derivative of f at the given values of x be

$$g_k = (1/n)\sum_j (d^k f/dx^k)_j = \sum_{i=k}^{N} a_i \binom{i}{k} k! C_{i-k}. \tag{5}$$

Let the quantity R be defined by

$$R^2 = \sum_{k=1}^{N} g_k^2/k! = \sum_{k=1}^{N}\Big[\sum_{i=k}^{N} a_i \binom{i}{k} k! C_{i-k}\Big]^2\Big/ k! \tag{6}$$

or

$$R^2 = \sum_{i,j,k=1}^{N} a_i a_j \binom{i}{k}\binom{j}{k} k! C_{i-k} C_{j-k}.$$

To establish the fact that $r^2 = R^2$ it is necessary to prove the identity

$$\sum_{k=1}^{N} \binom{i}{k}\binom{j}{k} k! C_{i-k} C_{j-k} = C_{i+j}. \tag{7}$$

When $i+j$ is odd, both members of (7) vanish. It remains to prove two identities for the cases in which i and j are both even or both odd:

$$\sum_k \binom{2i}{2k}\binom{2j}{2k} (2k)! C_{2i-2k} C_{2j-2k} = C_{2i+2j} \tag{7a}$$

and

$$\sum \binom{2i+1}{2k+1}\binom{2j+1}{2k+1} (2k+1)! C_{2i-2k} C_{2j-2k} = C_{2i+2j+2}. \tag{7b}$$

These last two equations reduce to

$$\sum_k \frac{(2i)!(2j)!2^{2k}}{(2k)!(i-k)!(j-k)!2^{i+j}} = \frac{(2i+2j)!}{2^{i+j}(i+j)!} \tag{8a}$$

and

$$\sum_k \frac{(2i+1)!(2j+1)!2^{2k+1}}{(2k+1)!(i-k)!(j-k)!2^{i+j+1}} = \frac{(2i+2j+2)!}{2^{i+j+1}(i+j+1)!}, \tag{8b}$$

respectively. Let us consider the coefficients of $x^{2i}y^{2j}$ in $(x^2+2xy+y^2)^{i+j}$ and in $(x+y)^{2i+2j}$; they are, respectively,

$$\sum_k \frac{(i+j)!2^{2k}}{(i-k)!(j-k)!(2k)!} \quad \text{and} \quad \frac{(2i+2j)!}{(2i)!(2j)!}; \tag{9a}$$

whence

$$\sum_k \left[\frac{(2i)!(2j)!2^{2k}}{(2k)!(i-k)!(j-k)!2^{i+j}}\right] = \frac{(2i+2j)!}{2^{i+j}(i+j)!}.$$

Similarly the coefficients of $x^{2i+1}y^{2j+1}$ in $(x^2+2xy+y^2)^{i+j+1}$ and in $(x+y)^{2i+2j+2}$ are, respectively,

$$\sum_k \frac{(i+j+1)!2^{2k+1}}{(2k+1)!(i-k)!(j-k)!} \quad \text{and} \quad \frac{(2i+2j+2)!}{(2i+1)!(2j+1)!}; \tag{9b}$$

whence

$$\sum_k \frac{(2i+1)!(2j+1)!2^{2k+1}}{(2k+1)!(i-k)!(j-k)!2^{i+j+1}} = \frac{(2i+2j+2)!}{2^{i+j+1}(i+j+1)!}.$$

Therefore the square of the correlation coefficient between y and $f(x)$ is equal to

$$r^2 = \sum_{i=1}^{N} [\bar{f}^{(i)}(x)]^2/i! \tag{10}$$

This expresses the square of this correlation coefficient as a function of the squares of the averages of the derivatives of the polynomial at the given values of x.

CORRECTION

L. Mirsky, *Diagonal elements of orthogonal matrices*, this MONHTLY, vol. 66, 1959, pp. 19–22. Professor Louis Brand has drawn attention to a misprint on page 20, line 4, where $-d_1, \cdots, -d_n$ should be changed to $-d_1, d_2, \cdots, d_n$.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

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RECURRENCE RELATIONS FOR SOLUTIONS OF PELL'S EQUATION

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It is well known that if x_1, y_1 be the smallest positive integral solution of the Pellian equation [1]:

$$x^2 - Dy^2 = 1, \tag{1}$$

where D is a positive integer which is not a square, then the general solution x_r, y_r in positive integers is given by

$$x_r + y_r\sqrt{D} = (x_1 + y_1\sqrt{D})^r, \tag{2}$$

where $r = 1, 2, \cdots$ [2]. The smallest solution x_1, y_1 of (1) is easily obtainable in terms of the continued fraction for $\sqrt{D}$ [3] but (2) is unwieldy for the computation of other solutions and it is more convenient to use recurrence relations which can readily be shown to follow from (2).

From (2) are obtained

$$x_{r+s} + y_{r+s}\sqrt{D} = (x_r + y_r\sqrt{D})(x_s + y_s\sqrt{D}), \tag{3}$$

and

$$x_{nr} + y_{nr}\sqrt{D} = (x_r + y_r\sqrt{D})^n. \tag{4}$$

Equating rational and irrational coefficients in (3) gives the general recurrence relations:

$$x_{r+s} = x_r x_s + Dy_r y_s, \qquad y_{r+s} = x_r y_s + y_r x_s. \tag{5}$$

Particular cases of (5) are

$$x_{r+1} = x_1 x_r + Dy_1 y_r, \qquad y_{r+1} = x_1 y_r + y_1 x_r; \tag{6}$$

$$x_{2r} = -1 + 2x_r^2, \qquad y_{2r} = 2x_r y_r; \tag{7}$$

$$x_{3r} = -3x_r + 4x_r^3, \qquad y_{3r} = (-1 + 4x_r^2)y_r. \tag{8}$$

Equation (6) enables one to go, with a minimum of computation, from x_1, y_1 to x_2, y_2 to x_3, y_3, and so on. Equations (7) and (8) are particular cases of general expressions for x_{nr} and y_{nr} and y_{nr}/y_r in terms of x_r, for $n = 1, 2, \cdots$. These are obtained by changing the notation in (1) and (4), either by putting $x_r = \cos\rho$, $y_r = \sin\rho$, $x_{nr} = \cos n\rho$, $y_{nr} = \sin n\rho$, $\sqrt{D} = \sqrt{-1} = i$, to get (9) and (10), respectively, or by putting $x_r = \cosh\rho$, $y_r = i\sinh\rho$, $x_{nr} = \cosh n\rho$, $y_{nr} = i\sin n\rho$, $\sqrt{D} = i$,

to get (11) and (12), respectively.

$$\cos^2 \rho + \sin^2 \rho = 1, \tag{9}$$

$$\cos n\rho + i \sin n\rho = (\cos \rho + i \sin \rho)^n, \tag{10}$$

$$\cosh^2 \rho - \sinh^2 \rho = 1, \tag{11}$$

$$\cosh n\rho - \sinh n\rho = (\cosh \rho - \sinh \rho)^n. \tag{12}$$

Hence it follows that x_{nr} can be expressed in terms of x_r in the same way that $\cos n\rho$ can be expressed in terms of $\cos \rho$, or that $\cosh n\rho$ can be expressed in terms of $\cosh \rho$, since these expressions are independent of D. Likewise, y_{nr}/y_r can be expressed in terms of x_r in the same way that $\sin n\rho/\sin \rho$ can be expressed in terms of $\cos \rho$. These expressions are well known [4]. The results are most conveniently expressed in the form:

$$\begin{aligned} x_{nr} &= 2^{n-1} x_r^n - \frac{n}{1!} 2^{n-3} x_r^{n-2} + \cdots \\ &\quad + (-1)^s \frac{n(n-s-1)\cdots(n-2s+1)}{s!} 2^{n-2s-1} x_r^{n-2s} + \cdots, \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{y_{nr}}{y_r} &= 2^{n-1} x_r^{n-1} - \frac{(n-2)}{1!} 2^{n-3} x_r^{n-3} + \cdots \\ &\quad + (-1)^s \frac{(n-s-1)\cdots(n-2s)}{s!} 2^{n-2s-1} x_r^{n-2s-1} + \cdots. \end{aligned} \tag{14}$$

Whereas (1) always has positive integral solutions the similar equation

$$u^2 - Dv^2 = -1, \tag{15}$$

is only soluble in positive integers u, v for values of D which satisfy certain other conditions [3]. In such cases, if u_1, v_1 is the smallest positive integral solution of (15), then all solutions are again given by an equation like (2). Consequently, equations like (3), (4), (5) and (6) also apply to solutions of (15), if they exist. Whereas, by (1), $Dy_r^2 = x_r^2 - 1$, it follows from (15) that $Dv_r^2 = u_r^2 + 1$, so that (7) and (8) now become, respectively:

$$u_{2r} = 1 + 2u_r^2, \qquad v_{2r} = 2u_r v_r; \tag{16}$$

$$u_{3r} = 3u_r + 4u_r^3, \qquad v_{3r} = (1 + 4u_r^2)v_r. \tag{17}$$

The general relations between u_{nr} or v_{nr}/v_r and u_r are

$$\begin{aligned} u_{nr} &= 2^{n-1} u_r^n + \frac{n}{1!} 2^{n-3} u_r^{n-2} + \cdots \\ &\quad + \frac{n(n-s-1)\cdots(n-2s+1)}{s!} 2^{n-2s-1} u_r^{n-2s} + \cdots, \end{aligned} \tag{18}$$

$$\frac{v_{nr}}{v_r} = 2^{n-1}u_r^{n-1} + \frac{(n-2)}{1!}2^{n-3}u_r^{n-3} + \cdots + \frac{(n-s-1)\cdots(n-2s)}{s!}2^{n-2s-1}u_r^{n-2s-1} + \cdots. \tag{19}$$

I am indebted to Professor R. L. Goodstein, who has previously given (7) [5], and who kindly pointed out to me that the value of x_{nr}, which I obtained otherwise, is the same polynomial in x_r that cosh $n\rho$ is in cosh ρ or that cos $n\rho$ is in cos ρ.

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NOTE ON AN IDENTITY OF CAYLEY*

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Cayley [1] proved an identity relating a third-order determinant and the corresponding permanent,

$$\begin{vmatrix} \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{b_1} & \frac{1}{b_2} & \frac{1}{b_3} \\ \frac{1}{c_1} & \frac{1}{c_2} & \frac{1}{c_3} \end{vmatrix}_+ \begin{vmatrix} \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{b_1} & \frac{1}{b_2} & \frac{1}{b_3} \\ \frac{1}{c_1} & \frac{1}{c_2} & \frac{1}{c_3} \end{vmatrix} = \begin{vmatrix} \frac{1}{a_1^2} & \frac{1}{a_2^2} & \frac{1}{a_3^2} \\ \frac{1}{b_1^2} & \frac{1}{b_2^2} & \frac{1}{b_3^2} \\ \frac{1}{c_1^2} & \frac{1}{c_2^2} & \frac{1}{c_3^2} \end{vmatrix}_+ \tag{1}$$

provided that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0, \tag{2}$$

and no element in the determinant (2) is zero.

In (1) the first factor on the left is a permanent of order 3, the general definition of a permanent of order n being

$$|a_{ij}|_+ = \sum a_{1p}a_{2q}\cdots a_{nr}, \tag{3}$$

* Work done under Office of Naval Research Contract Nonr 870(00).

where the sum on the right is over all $n!$ permutations $pq \cdots r$ of $1\ 2 \cdots n$, (Muir [2]). The expansion of the permanent (3) thus follows that of a determinant except all signs are taken as plus.

In this note we show that identity (1) can be extended to the corresponding products of the fourth order according to the theorem:

THEOREM. *If* $|a_{ij}|$ *is a determinant of order* $n \leqq 4$, *and if the rank of the matrix* $[a_{ij}]$ *is less than* 3 (*and all* $a_{ij} \neq 0$), *then*

$$\left|\frac{1}{a_{ij}}\right|^{+}\left|\frac{1}{a_{ij}}\right|^{+} = \left|\frac{1}{a_{ij}^2}\right|. \tag{4}$$

The theorem is obviously true if $n=1$ or 2, and the case $n=3$ is Cayley's result (1). There remains the case $n=4$. In this case we make use of the formula of Muir [2] giving the product of a general permanent and determinant,

$$\left|\begin{matrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{nn} \end{matrix}\right|^{+}\left|\begin{matrix} x_{11} & \cdots & x_{1n} \\ x_{21} & \cdots & x_{2n} \\ \cdot & \cdots & \cdot \\ a_{n1} & \cdots & x_{nn} \end{matrix}\right|^{+} = \sum \pm \left|\begin{matrix} a_{11}x_{i1} & \cdots & a_{1n}x_{in} \\ a_{21}x_{j1} & \cdots & a_{2n}x_{jn} \\ \cdot & \cdots & \cdot \\ a_{n1}x_{k1} & \cdots & a_{nn}x_{kn} \end{matrix}\right|, \tag{5}$$

the summation being over all $n!$ permutations $ij \cdots k$ of $1\ 2 \cdots n$, and the $+$ and $-$ sign is used according as $ij \cdots k$ is an even or odd permutation of $1\ 2 \cdots n$ respectively.

Now in (5), (with $n=4$), put $x_{ij}=a_{ij}$, and then replace a_{ij} by $1/a_{ij}$ (assuming all $a_{ij} \neq 0$). Then the 24 terms on the right side of (5) can be combined into five terms, as follows:

$$\left|\frac{1}{a_{ij}}\right|^{+}\left|\frac{1}{a_{ij}}\right|^{+} = [abcd] + 2[adbc] + 2[dbac] + 2[dacb] + 2[cabd], \tag{6}$$

where

$$[pqrs] \equiv \left|\begin{matrix} \frac{1}{a_1p_1} & \frac{1}{a_2p_2} & \frac{1}{a_3p_3} & \frac{1}{a_4p_4} \\ \frac{1}{b_1q_1} & \frac{1}{b_2q_2} & \frac{1}{b_3q_3} & \frac{1}{b_4q_4} \\ \frac{1}{c_1r_1} & \frac{1}{c_2r_2} & \frac{1}{c_3r_3} & \frac{1}{c_4r_4} \\ \frac{1}{d_1s_1} & \frac{1}{d_2s_2} & \frac{1}{d_3s_3} & \frac{1}{d_4s_4} \end{matrix}\right|,$$

and on the right side of (6) we have placed $a_j=a_{1j}$, $b_j=a_{2j}$, $c_j=a_{3j}$, $d_j=a_{4j}$.

The first determinant $[abcd]$ on the right side of (6) is the same as the determinant on the right side of (4). The other four determinants on the right side of (6) are all zero on making use of the rank assumption of matrix $[a_{ij}]$.

To see this, consider, for example, the determinant $[adbc]$. This can be expressed as

$$(7)\qquad [abdc]=\begin{vmatrix} \frac{1}{a_1^2} & \frac{1}{a_2^2} & \frac{1}{a_3^2} & \frac{1}{a_4^2} \\ \frac{1}{b_1d_1} & \frac{1}{b_2d_2} & \frac{1}{b_3d_3} & \frac{1}{b_4d_4} \\ \frac{1}{c_1b_1} & \frac{1}{c_2b_2} & \frac{1}{c_3b_3} & \frac{1}{c_4b_4} \\ \frac{1}{d_1c_1} & \frac{1}{d_2c_2} & \frac{1}{d_3c_3} & \frac{1}{d_4c_4} \end{vmatrix} = A \begin{vmatrix} \frac{b_1c_1d_1}{a_1^2} & \frac{b_2c_2d_2}{a_2^2} & \frac{b_3c_3d_3}{a_3^2} & \frac{b_4c_4d_4}{a_4^2} \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix},$$

where $A=(b_1c_1d_1)^{-1}(b_2c_2d_2)^{-1}(b_3c_3d_3)^{-1}(b_4c_4d_4)^{-1}$.

By expanding the second determinant in (7) by the elements of its first row, it follows that $[adbc]=0$ since we are assuming that matrix

$$[a_{ij}]=\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

is of rank less than 3.

A similar proof shows $[dbac]=[dacb]=[cabd]=0$. The proof of the theorem now follows since as previously stated the remaining determinant on the right side of (6) $[abcd]=|1/a_{ij}^2|$.

As an illustration of the theorem we may take $|a_{ij}|$ to be

$$|a_{ij}|=\begin{vmatrix} a & a+d & a+2d & a+3d \\ a+4d & a+5d & a+6d & a+7d \\ a+8d & a+9d & a+10d & a+11d \\ a+12d & a+13d & a+14d & a+15d \end{vmatrix}.$$

The author thanks the referee for his helpful suggestions.

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ON THE REDUCIBILITY OF SOME LINEAR DIFFERENTIAL OPERATORS*

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The differential equations

$$[x^nD^{2n} - 1]y = 0, \tag{1}$$

$$[x^{2n}D^n - 1]y = 0, \tag{2}$$

have been solved previously by Lommel [1a] and Steen [2a], respectively, but not in a direct manner. By showing that the operators x^nD^{2n} and $x^{2n}D^n$ are reducible, we immediately obtain the solutions of (1) and (2) in terms of solutions of a second-order and a first-order equation, respectively. We also obtain reducible generalizations of the above operators.

The linear polynomial operator $P(D, x)$ is defined to be reducible if it can be expressed in the form

$$P(D, x) \equiv F[G(D, x)], \tag{3}$$

where $G(D, x)$ is a linear polynomial operator of order less than that of $P(D, x)$ (F will then have to be a polynomial of degree $\geqq 2$). It then follows that the solution to the equation $P(D, x)y = ay$ is given by the solution of the lower order equation $G(D, x)y = \lambda y$, where $F(\lambda) = a$.

We now show that the operator x^nD^{2n} is reducible to

$$x^nD^{2n} \equiv [xD^2 - (n-1)D]^n. \tag{4}$$

Letting $x = e^z$ and $xD = \theta = d/dz$, it follows that

$$x^nD^{2n} = e^{-nz}\theta(\theta - 1)\cdots(\theta - 2n + 1), \tag{5}$$

$$[xD^2 - (n-1)D]^n = [e^{-z}\theta(\theta - n)]^n = e^{-z}\theta(\theta - n)e^{-z}\theta(\theta - n)\cdots e^{-z}\theta(\theta - n). \tag{6}$$

By using the exponential shift theorem

$$F(\theta)e^{\int\psi dz} = e^{\int\psi dz}F(\theta + \psi),$$

we can transform (6) into (5) by shifting all the e^{-z} terms to the extreme left.

The solution to (1) is now immediately given by the solution of the second order equation

$$[xD^2 - (n-1)D]y = \lambda_r y, \tag{7}$$

where $\{\lambda_r\}$ are the n roots of $\lambda^n = 1$. Equation (7) is a modified† Bessel equation whose solution is $y = x^{n/2}Z_n(2\sqrt{-\lambda_r x})$. Hence, the general solution of (1) is

* Presented to the American Mathematical Society, August 29, 1957.

† We have used the term *modified* Bessel equation to apply to any second order equation whose solution can be expressed "simply" in terms of Bessel functions other than $AJ_\nu(x) + BY_\nu(x)$.

$$(8)\qquad y = x^{n/2}\sum_{r=0}^{n-1}\left\{A_r J_n(2\sqrt{-\lambda_r x}) + B_r Y_n(2\sqrt{-\lambda_r x})\right\}.$$

Another application is in solving the Nicholson [1b] type equation

$$D^2 z^4 D^2 u = z^2 u.$$

Since $D^m z^m$ and $z^n D^n$ commute, the equation can be rewritten as

$$(z^2 D^4 - 1)z^2 u = 0,$$

and since $z^2 D^4$ is reducible, the equation is easily solved.

A dual identity to (4) is given by

$$(9)\qquad x^{2n} D^n \equiv [x^2 D - (n-1)x]^n$$

which can be established in a similar manner. Consequently, the general solution of (2) is

$$(10)\qquad y = \sum_{r=1}^{n} A_r x^{n-1} e^{-\lambda_r/x}$$

where $\{\lambda_r\}$ are the n roots of $\lambda^n = 1$.

As an interesting aside, it follows from our two identities that two noncommutative solutions of the operator equation $A^n B^n C^n = (ABC)^n$ are $A = x^{-1}$, $B = x^2 D - (n-1)x$, $C = D$, or $A = x$, $B = xD^2 - (n-1)D$, $C = D^{-1}$.

Next we will show that the identity

$$(11)\qquad z^{n+1}\left(\frac{1}{z}\frac{d}{dz}\right)^n \equiv \frac{1}{z^{n+1}}\left[\left(z^3\frac{d}{dz}\right)^n z^{2-2n}\right];$$

derived by Glaisher [1c], is equivalent to identity (9). If we let $x = 1/z^2$, (11) is transformed into $x^{2n} D^n \equiv x^{n-1}(x^2 D)^n x^{1-n}$.

By the exponential shift theorem, $x^{n-1}(x^2 D)^n x^{1-n} \equiv [x^2 D - (n-1)x]^n$.

Generalizations of the previous identities are given in the following four equations:

$$(12)\quad x^{rn} D^{(r+1)n} \equiv [(xD + 1 - n)(xD + 1 - 2n) \cdots (xD + 1 - rn) D]^n,$$

$$(13)\quad x^{(r+1)n} D^{rn} \equiv [x(xD + 1 - n)(xD + 1 - 2n) \cdots (xD + 1 - rn)]^n,$$

$$(14)\quad D^{(r+1)n} x^{rn} \equiv [(xD + 1 + n)(xD + 1 + 2n) \cdots (xD + 1 + rn) D]^n,$$

$$(15)\quad D^{rn} x^{(r+1)n} \equiv [x(xD + 1 + n)(xD + 1 + 2n) \cdots (xD + 1 + rn)]^n.$$

We can now solve the equation

$$(16)\qquad [x^{3n} D^{2n} - 1]y = 0,$$

in terms of Bessel functions. From (13) it follows that the solution is given by the solution of

$$x(xD + 1 - n)(xD + 1 - 2n)y = \lambda y,$$

where $\lambda^n = 1$. This latter equation is the modified Bessel equation [2b]

$$\left[x^2D^2 + 3(1-n)xD + (1-n)(1-2n) - \frac{\lambda}{x}\right]y = 0,$$

whose solution is $y = x^{(3n/2)-1}Z_n\{2\sqrt{-\lambda/x}\}$.

Remark. It would be of interest to find necessary and/or sufficient conditions for a linear differential operator to be irreducible.

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A REMARK ON ENNEPER'S MINIMAL SURFACE*

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In a paper of 1864 A. Enneper wrote down the formulas of a special minimal surface,

$$S: \begin{cases} x = u + uv^2 - \frac{1}{3}u^3 \\ y = v + u^2v - \frac{1}{3}v^3 \\ z = u^2 - v^2 \end{cases} \tag{1}$$

which, in the sequel, has been extensively studied in the literature and used for illustrative purposes.† Equations (1) give the parametric representation. However, after elimination of the parameters u and v, S also can be recognized as an algebraic surface of ninth degree:

$$\left[y^2 - x^2 + \frac{4}{3}z + \frac{4}{9}z^3\right]^3 = 3z^2\left[y^2 - x^2 + \frac{8}{9}z - z\left(x^2 + y^2 + \frac{8}{9}z^2\right)\right]^2. \tag{2}$$

The surface contains the origin $x=y=z=0$, and its normal there is parallel to the z-axis. Consequently, it is possible, in a neighborhood of the origin, to bring the equation of S into the form $z=z(x, y)$. We are concerned in the present note with the determination of a circle in the xy-plane in which this special representation can be maintained. More precisely, we shall prove the following assertion:

The part of Enneper's surface containing the origin can be brought into the form $z=z(x, y)$ *over the circle* $x^2+y^2<(64/243)$.

An interesting conclusion can be drawn from this result. Suppose a minimal

* This paper was prepared under Contract N onr—710(16) between the University of Minnesota and the Office of Naval Research.

† See for instance A. Enneper [3], pp. 107–108; G. Darboux [1], pp. 316–320; T. Radó [7], p. 40; L. P. Eisenhart [2].

surface $z=z(x, y)$ is defined over the circle $x^2+y^2<R^2$. Denote its Gaussian curvature at $x=y=0$ by K_0. According to an inequality of E. Heinz* $|K_0| \leqq C\cdot R^{-2}W_0^{-2}(W=\sqrt{1+z_x^2+z_y^2})$. Here C is a universal constant. Its value is unknown. The best estimate so far is $C\leqq 49/4$, see [6]. It is easy to compute for Enneper's surface the values $K_0=-4$, $W_0=1$. Thus the following estimate from below for C holds: $C\geqq(256/243)=1.05$.

In order now to prove our assertion we have to investigate the mapping $u, v\to x, y$, defined by the first two equations of (1). In domains where this mapping is one-to-one, u and v can be expressed in terms of x, y and substituted into the last equation of (1). Introducing polar coordinates $u \quad \rho\cos\theta$, $v=\rho\sin\theta$ we obtain

$$x = a(\rho,\theta)\cdot\cos\theta, \qquad a(\rho,\theta) = \rho\left(1+\rho^2-\frac{4}{3}\rho^2\cos^2\theta\right);$$

(4)

$$y = b(\rho,\theta)\cdot\sin\theta, \qquad b(\rho,\theta) = \rho\left(1+\rho^2-\frac{4}{3}\rho^2\sin^2\theta\right).$$

Denote by C_ρ the image curve in the xy-plane of the circle $u^2+v^2=\rho^2$. Its curvature, up to a positive factor, is given by the expression $x_\theta y_{\theta\theta}-x_{\theta\theta}y_\theta$. A straightforward computation yields

(5)
$$\begin{aligned} x_\theta y_{\theta\theta}-x_{\theta\theta}y_\theta &= \rho^2(1-2\rho^2\cos 4\theta-3\rho^4) \\ &\geqq \rho^2(1-2\rho^2-3\rho^4) = \rho^2(1+\rho^2)(1-3\rho^2). \end{aligned}$$

Hence the curves C_ρ are convex as long as $\rho<1/\sqrt{3}$. The following method—a "generalized ellipse construction"—for the construction of the curves C_ρ is suggested. First draw, for fixed ρ, the two curves

(6)
$$A_\rho:\begin{cases} x = a(\rho,\theta)\cdot\cos\theta \\ y = a(\rho,\theta)\cdot\sin\theta \end{cases}; \qquad B_\rho:\begin{cases} x = b(\rho,\theta)\cdot\cos\theta \\ y = b(\rho,\theta)\cdot\sin\theta \end{cases}.$$

B_ρ can be obtained by rotation of A_ρ about an angle of 90 degrees. The radius vector in the xy-plane which forms the angle θ with the x-axis intersects A_ρ in a point P_ρ and B_ρ in a point Q_ρ. The new point whose abscissa is that of P_ρ and whose ordinate is that of Q_ρ will be a point on C_ρ.

We now proceed to prove two properties of the curve A_ρ (and, *a fortiori*, B_ρ):

1. The curve A_ρ is convex for $\rho<1/\sqrt{3}$. The curvature of A_ρ, up to a positive factor, is given by $x_\theta y_{\theta\theta}-x_{\theta\theta}y_\theta=a^2+2a_\theta^2-aa_{\theta\theta}$. Substituting the function $a(\rho,\theta)$ from (4_1), we obtain

$$(1+\rho^2)\left(1+\frac{11}{3}\rho^2\right)+\frac{8}{3}\rho^2(\rho^2-3)\cos^2\theta-\frac{16}{3}\rho^4\cos^4\theta,$$

* See E. Heinz [4]; E. Hopf [5]; J. Nitsche [6].

an expression which, as long as $\rho^2-3\leqq 0$, is not smaller than

$$(1+\rho^2)\left(1+\frac{11}{3}\rho^2\right)+\frac{8}{3}\rho^2(\rho^2-3)-\frac{16}{3}\rho^4=\left(\frac{1}{3}-\rho^2\right)(3-\rho^2).$$

Therefore the curvature of A_ρ is positive for $\rho<1/\sqrt{3}$.

2. A_{ρ_1} lies inside A_{ρ_2} for $\rho_1<\rho_2<1/\sqrt{3}$. In order to show this we observe for one thing that

$$a(\rho,\theta)\geqq\rho\left(1+\rho^2-\frac{4}{3}\rho^2\right)=\rho\left(1-\frac{1}{3}\rho^2\right)>0 \quad \text{for } \rho<1/\sqrt{3}.$$

On the other hand we have

$$a_\rho(\rho,\theta)=1+3\rho^2-4\rho^2\cos^2\theta\geqq 1-\rho^2>0 \qquad \text{for } \rho<1.$$

From the properties ascertained above it follows that the curves C_ρ form a nested family of convex curves for $\rho<1/\sqrt{3}$. Furthermore C_ρ does not enter the interior of the intersection of the interior of A_ρ and that of B_ρ. Hence we can conclude that the circle $\sqrt{x^2+y^2}<\rho(1+\rho^2-4\rho^2/3)=\rho(1-\rho^2/3)$ is contained in the interior of C_ρ. Our statement follows for $\rho\to 1/\sqrt{3}$.

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CURVE-FITTING MATRICES

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In a recent note, Wilf [1] explicitly calculated inverses required for polynomial curve-fitting up through the case of 6 points. For machine usage, it would seem desirable to have readily calculable inverses for arbitrary order n. One method for doing this was demonstrated by Macon and Spitzbart [2]. Another procedure for accomplishing this is indicated below.

Using a notation similar to that of Wilf, we desire a polynomial

$$p(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}$$

such that

$$p(x_i) = y_i, \qquad i = 1, \cdots, n.$$

This reduces to the matrix form $V_n a = y$, where $a' = (a_0 a_1 \cdots a_{n-1})$, $y' = (y_1 y_2 \cdots y_n)$, and the ith row of V_n is $(1\ x_i \cdots x_i^{n-1})$. After computing the inverse of V_n, the problem is solved by forming $a = V_n^{-1} y$.

Let

$$f(x) \equiv (x + x_1) \cdots (x + x_n) = x^n - s_1 x^{n-1} + \cdots + (-1)^n s_n.$$

The numbers s_k can be generated recursively by calculating

$$s_{i,j} = s_{i,j-1} - x_j s_{i-1,j-1}, \qquad i = 1, \cdots, j,$$

letting j run from 1 to n, and defining $s_{0,j-1} = 1$, $s_{j,j-1} = 0$, where $s_k = s_{k,n}$. The quantity s_n is not required. Thus

$$s_{1,1} = s_{1,0} - x_1 s_{0,0} = 0 - x_1(1) = -x_1,$$

$$s_{1,2} = s_{1,1} - x_2 s_{0,1} = -x_1 - x_2(1) = -x_1 - x_2,$$
$$s_{2,2} = s_{2,1} - x_2 s_{1,1} = 0 - x_2(-x_1) = x_1 x_2,$$

$$s_{1,3} = s_{1,2} - x_3 s_{0,2} = -x_1 - x_2 - x_3(1),$$
$$s_{2,3} = s_{2,2} - x_3 s_{1,2} = x_1 x_2 - x_3(-x_1 - x_2),$$
$$s_{3,3} = s_{3,2} - x_3 s_{2,2} = 0 - x_3(x_1 x_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

To compute the elements appearing in the successive columns of the inverse, calculate

$$q_{i,j} = s_i + x_j q_{i-1,j}, \qquad r_{i,j} = q_{i,j} + x_j r_{i-1,j},$$

$i = 1, \cdots, n-1$, where j runs from 1 to n, and $q_{0,j}$ and $r_{0,j}$ are defined to be 1. In expanded form

$$q_{1,j} = s_1 + x_j q_{0,j} = s_1 + x_j,$$
$$q_{2,j} = s_2 + x_j q_{1,j} = s_2 + x_j s_1 + x_j^2,$$
$$q_{3,j} = s_3 + x_j q_{2,j} = s_3 + x_j s_2 + x_j^2 s_1 + x_j^3,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$r_{1,j} = q_{1,j} + x_j r_{0,j} = s_1 + 2x_j,$$
$$r_{2,j} = q_{2,j} + x_j r_{1,j} = s_2 + 2x_j s_1 + 3x_j^2,$$
$$r_{3,j} = q_{3,j} + x_j r_{2,j} = s_3 + 2x_j s_2 + 3x_j^2 s_1 + 4x_j^3,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Finally, defining d_j as the reciprocal of $r_{n-1,j}$, the jth column of the inverse is the transpose of the vector $(d_j q_{n-1,j} d_j q_{n-2,j} \cdots d_j q_{1,j} d_j)$.

If we multiply the resulting matrix on the left by the matrix whose ith row

is $(1\ x_i \cdots x_i^{n-2} x_i^{n-1})$, the element appearing in the ith row and jth column of the product is

$$\begin{aligned} & d_j(q_{n-1,j} + x_i q_{n-2,j} + \cdots + x_i^{n-2} q_{1,j} + x_i^{n-1}) \\ &= d_j[s_{n-1} + x_j s_{n-2} + \cdots + x_j^{n-2} s_1 + x_j^{n-1} \\ &\qquad + x_i s_{n-2} + \cdots + x_i x_j^{n-3} s_1 + x_i x_j^{n-2} \\ &\qquad\qquad + \cdots \\ &\qquad\qquad\qquad + x_i^{n-2} s_1 + x_i^{n-2} x_j \\ &\qquad\qquad\qquad\qquad + x_i^{n-1}] \\ &= \begin{cases} d_j(-1)^{n-1} \dfrac{f(-x_j) - f(-x_i)}{(-x_j) - (-x_i)} = 0, & j \neq i, \\ d_i(-1)^{n-1} f'(-x_i) = d_i \sigma_{n-1,i} = 1, & j = i, \end{cases} \end{aligned}$$

since

$$\begin{aligned} 0 = f(-x_i) &= (-1)^n (x_i^n + s_1 x_i^{n-1} + \cdots + s_{n-1} x_i + s_n), \\ 0 = f(-x_j) &= (-1)^n (x_j^n + s_1 x_j^{n-1} + \cdots + s_{n-1} x_j + s_n), \\ 0 \neq (-1)^{n-1} f'(-x_i) &= n x_i^{n-1} + (n-1) s_1 x_i^{n-2} + \cdots + s_{n-1}. \end{aligned}$$

Evidently, all the identities involved in the above discussion could be proved by induction.

The referee suggested by way of contrast the following procedure. Assuming the form

$$p(x) = a_0 + a_1(x - x_1) + \cdots + a_{n-1}(x - x_1)^{n-1},$$

we observe first that

$$a_i = p^{(i)}(x_1)/i!. \tag{1}$$

Also, since Lagrangian interpolation is exact here, we may write

$$p(x) = \sum_j \frac{p_n(x)}{(x - x_j) p_n'(x_j)} y_j,$$

where $p_n(x) = \prod_k (x - x_k)$. Thus

$$p'(x) = \sum_j \frac{y_j}{p_n'(x_j)} \frac{(x - x_j) p_n'(x) - p_n(x)}{(x - x_j)^2}$$

and

$$y_i' \equiv p'(x_i) = \sum_j C_{ij} y_j, \tag{2}$$

where

$$C_{ij} = \begin{cases} \dfrac{p_n'(x_i)}{(x_i - x_j)p_n'(x_j)}, & i \neq j, \\ \dfrac{p_n''(x_j)}{2p_n'(x_j)}, & i = j. \end{cases}$$

Since the quantities C_{ij} depend only on the x's, relation (2) continues to hold if we replace y_i by y_i', and y_i' by y_i''. Thus $y' = Cy$, $y'' = Cy' = C^2y$, *etc.* Comparing this information with (1), it follows that the $(k+1)$st row of the desired inverse, the one to obtain the a's for the present case, is the first row of the kth power of the matrix C divided by $k!$. The first row, by inspection, has a one in the first position and zeros elsewhere.

Although this procedure is conceptually interesting from a computational standpoint, it appears that the number of steps involved in computing C and C^2 already exceeds the total number required in the method originally discussed above.

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CLASSROOM NOTES

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ON THE METHOD OF VARIATION OF PARAMETERS

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Given the linear differential equation of second order,

$$y'' + P(x)y' + Q(x)y = R(x), \tag{1}$$

let the complementary solution, y_c, be given by $y_c = C_1u + C_2v$, where u and v are two linearly independent solutions of (1) with $R \equiv 0$. Let the particular integral, y_p, be given by $y_p = Au + Bv$, where A and B are functions of x to be determined. Then the variation of parameters method yields the system of two equations,

$$A'u + B'v = 0, \tag{2}$$

$$A'u' + B'v' = R. \tag{3}$$

Since the Wronskian, $W = uv' - vu' \neq 0$, we find that

$$y_p = -u\int \frac{vR}{W}\,dx + v\int \frac{uR}{W}\,dx. \tag{4}$$

It is recalled that (2) is arbitrary, that (3) is determined from (1), and that the general solution of (1) is $y=y_c+y_p$. We note that (2) and (3) remain the same if $y_p=(A+\alpha)u+(B+\beta)v$, where α and β are *specified* constants. In practice, one chooses $\alpha=\beta=0$.

Let $f=f(x, u, v)$ be a function of class C^1 in x. If we introduce f into the right-hand side of (2), then (3) assumes a new form. Assuming a particular integral, $Y_p=(A+\alpha)u+(B+\beta)v$, where α and β are constants whose values will be specified later, the new system of equation is

$$A'u + B'v = f, \qquad A'u' + B'v' = R - f' - Pf. \tag{5}$$

We will now show that (5), accompanied with a proper specification of α and β, yields the particular integral $Y_p\equiv y_p$, as given by (4).

We find that $WA'=-vR+vf'+fv'+Pvf$, $WB'=uR-(uf'+fu'+Puf)$, $Y_p=y_p+g(x)$, where

$$g(x) = u\int \frac{Pvf+(vf)'}{W}\,dx - v\int \frac{Puf+(uf)'}{W}\,dx + \alpha u + \beta v. \tag{6}$$

Given f, there exists suitable choices of α and β such that $g(x)\equiv 0$. Suppose $f\equiv 0$. Then, if $\alpha=\beta=0$, $g(x)\equiv 0$. We show now that there exists no $f\not\equiv 0$ such that $Pvf+(vf)'\equiv 0$ and $Puf+(uf)'\equiv 0$ simultaneously. Suppose now that such an $f\not\equiv 0$ exists. Then, since $W'+PW\equiv 0$, we have $(vf)'W-W'vf\equiv 0$ and $(uf)'W-W'uf\equiv 0$; thus $vf\equiv k_1W$ and $uf\equiv k_2W$, where $k_1\not\equiv 0$, $k_2\not\equiv 0$ are fixed constants. From this, we have that $k_1u\equiv k_2v$, which contradicts the assumption that u and v are linearly independent solutions of (1). Thus, given $f\not\equiv 0$, only one of the following three conditions must hold: (i) $vf\equiv k_1W$, $k_1\not\equiv 0$; (ii) $uf\equiv k_2W$, $k_2\not\equiv 0$; or (iii) neither (i) nor (ii) holds.

Let $f=K_1h+K_2q+K_3r$, where the K_i are constants, of which none, some, or all may be zero, and where $h=h(x, u, v)\not\equiv 0$, $q=q(x, u, v)\not\equiv 0$, and $r=r(x, u, v)=\sum_{i=1}^{N} r_i(x, u, v)\not\equiv 0$, $r_i\not\equiv 0$, are functions of class C^1 in x satisfying the following conditions: $vh\equiv aW$, $uq\equiv bW$, and $v\sum r_i\not\equiv a_\nu W$, $u\sum r_i\not\equiv b_\nu W$, $\nu=1, 2, \cdots, 2^N-1$ where $a\not\equiv 0$, $b\not\equiv 0$, $a_\nu\not\equiv 0$, and $b_\nu\not\equiv 0$ are constants, and where $\sum$ indicates summation over the set s_ν which is always a nonempty subset of the set of integers, $(1, \cdots, N)$, and which is one of the (2^N-1) sets determined by enumerating the combinations, $\binom{N}{k}$, $k=1, \cdots, N$. (For $N=3$, there are seven sets, s_ν: (1), (2), (3), (1, 2), (1, 3), (2, 3), (1, 2, 3)). Noting that if $\sigma\equiv q$ or $\sigma\equiv r$, then $Pv\sigma+(v\sigma)'\not\equiv 0$, and that if $\delta\equiv h$ or $\delta\equiv r$, then $Pu\delta+(u\delta)'\not\equiv 0$, we find, using the relation, $W'+PW\equiv 0$, that

$$\int \frac{Pv\sigma+(v\sigma)'}{W}\,dx = \frac{v\sigma}{W}, \qquad \int \frac{Pu\delta+(u\delta)'}{W}\,dx = \frac{u\delta}{W}. \tag{7}$$

Since $vh \equiv aW$ and $uq \equiv bW$, we have $Pvh + (vh)' \equiv 0$ and $Puq + (uq)' \equiv 0$. Using these relations and (7), we find that

$$u\int \frac{Pvf + (vf)'}{W}\,dx = uK_2\int \frac{Pvq + (vq)'}{W}\,dx + uK_3\int \frac{Pvr + (vr)'}{W}\,dx \tag{8}$$
$$= (K_2uqv)/W + (K_3uvr)/W = bvK_2 + (K_3uvr)/W,$$

$$v\int \frac{Puf + (uf)'}{W}\,dx = vK_1\int \frac{Puh + (uh)'}{W}\,dx + vK_3\int \frac{Pur + (ur)'}{W}\,dx \tag{9}$$
$$= (K_1vhu)/W + (K_3uvr)/W = auK_1 + (K_3uvr)/W.$$

Substitution of (8) and (9) into (6) yields $g(x) = (\alpha - aK_1)u + (\beta + bK_2)v$. Thus, if $\alpha = aK_1$, $\beta = -bK_2$, then $g(x) \equiv 0$. This completes the proof. Since each constant K_i has two states, $K_i = 0$ or $K_i \neq 0$, the function f may occur in $2^3 = 8$ ways. In only two cases, when $f \equiv 0$ or $f = K_3r$, can one choose $\alpha = \beta = 0$.

SOLIDS OF REVOLUTION

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Except for an occasional isolated example, most calculus books have no discussion of solids generated by the rotation of an area about a line parallel to neither axis. This note attempts to summarize in a brief, informal way, some techniques for determining such volumes.

It will be assumed that the area to be rotated can be summed in terms of the usual elements in rectangular and polar coordinates, and that the given line, $ax + by + c = 0$, (in polar coordinates, $r\cos(\theta - \omega) - p = 0$), lies outside the area, although it may have certain contacts of even order with it.

Given the centroid, $(\overline{X}, \overline{Y})$, (in polar coordinates, $(\overline{R}, \overline{\Theta})$), of an element of area, and its perpendicular distance, D, to a line, it follows from the Theorem of Pappus that the elementary volume of rotation is $2\pi DdA$. In the following table are listed centroids and distances for the basic elements of area.

Element of Area	Centroid	Distance to Line
ydx	$(\overline{X}, \overline{Y}) = \left[x, \frac{y_1(x) + y_2(x)}{2}\right]$	
xdy	$(\overline{X}, \overline{Y}) = \left[\frac{x_1(y) + x_2(y)}{2}, y\right]$	$D = \lambda\overline{X} + \mu\overline{Y} - p$
$dxdy$	$(\overline{X}, \overline{Y}) = (x, y)$	
$\frac{1}{2}(r_1^2 - r_2^2)d\theta$	$(\overline{R}, \overline{\Theta}) = \left[\frac{2}{3}\frac{r_1^2(\theta) + r_1(\theta)r_2(\theta) + r_2^2(\theta)}{r_1(\theta) + r_2(\theta)}, \theta\right]$	$D = \overline{R}\cos(\overline{\Theta} - \omega) - p$
$rdrd\theta$	$(\overline{R}, \overline{\Theta}) = (r, \theta)$	

INDEPENDENCE OF PATH FOR LINE INTEGRALS

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The geometric methods of Lovitt [1] may be extended to provide a proof of the equivalence of $\partial P/\partial y$ and $\partial Q/\partial x$ as a necessary and sufficient condition for the line integral $\int P(x, y)dx+Q(x, y)dy$ to be independent of path in a simply connected domain without the aid of Green's theorem. In addition, an interesting interpretation of the standard line integral formulas for the area enclosed by a plane curve follows immediately from another look at a seldom-mentioned geometrical interpretation of line integrals in the plane.

Throughout the discussion let C be a piecewise smooth curve in the XY-plane; *i.e.*, C consists of a finite number of arcs, each representable in parametric form by a pair of continuous functions having continuous derivatives for the range of the parameter. Moreover, we shall assume that $P(x, y)$ and $Q(x, y)$ are continuous functions having continuous first partial derivatives in a domain D containing C. Now, as indicated in Figure 1, let C_P and C_Q denote the curves of

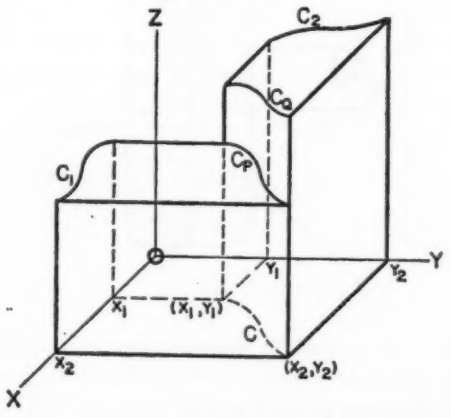

FIG. 1

intersection of the cylinder generated by C, with elements parallel to the Z-axis, with the surfaces $z=P(x, y)$ and $z=Q(x, y)$, respectively. The projection of C_P onto the XZ-plane will be designated by C_1, while C_2 will denote the corresponding projection of C_Q onto the YZ-plane. Then $\int_C P(x, y)dx+Q(x, y)dy$ from (x_1, y_1) to (x_2, y_2) is equal to the algebraic sum of the "areas under the curves" C_1 from x_1 to x_2 and C_2 from y_1 to y_2. The appropriate sign to be associated with each area is the conventional one that is employed as regards areas above or below the axes and increasing or decreasing end points of integration. (Mention of the above interpretation appears, for example, in [3], p. 49).

If C is extended to be a simple closed curve in the XY-plane, then the closed

line integral $\oint_C P dx + Q dy$ can be interpreted as the algebraic sum of the areas of two regions, one R_1 in the XZ-plane and the other R_2 in the YZ-plane, formed by projecting the curves of intersection of the cylindrical surface generated by C, with elements parallel to the Z-axis, with the surfaces $z = P(x, y)$ and $z = Q(x, y)$, respectively. Employing a right-handed coordinate system, let the positive direction of curves be counterclockwise in the XY- and XZ-planes and clockwise in the YZ-plane when viewed from the first octant, as suggested in Figure 2 by the curved arrows in each coordinate plane near the origin. (For a precise statement on direction orientation see [2], p. 232). Consequently, as we pass through a point (x_0, y_0) on C, C and C_1 will be traversed by corresponding

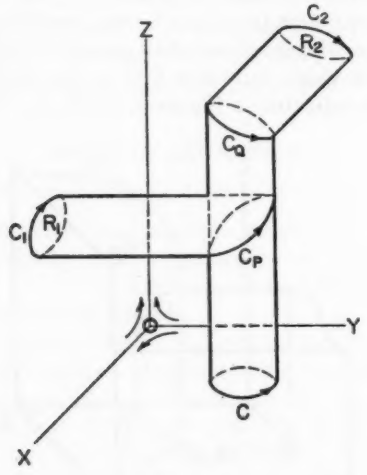

FIG. 2

points in the same direction sense if $\partial P/\partial y < 0$ at (x_0, y_0), and in the opposite sense if $\partial P/\partial y > 0$ at (x_0, y_0). On the other hand, C and C_2 will be traversed by corresponding points in the same sense if $\partial Q/\partial x > 0$, and in the opposite sense if $\partial Q/\partial x < 0$, at a point on C as we pass through that point. In particular, if A_1 and A_2 denote the areas of regions R_1 and R_2, respectively, the counterclockwise line integral $\oint_C P dx = -(\operatorname{sgn} \partial P/\partial y) A_1$ if $\partial P/\partial y$ maintains the same sign on C, while the counterclockwise line integral $\oint_C Q dy = (\operatorname{sgn} \partial Q/\partial x) A_2$ if $\partial Q/\partial x$ maintains the same sign on C.

Now, if $P(x, y) \equiv y$ and $Q(x, y) \equiv x$ it follows from symmetry that the areas A_1 and A_2 of the regions enclosed by C_1 and C_2 will each be equal to the area of the region enclosed by C. Hence, the area of the region enclosed by C will be given by the counterclockwise line integrals $-\oint_C y dx$ or $\oint_C x dy$; and, therefore, also by $\frac{1}{2}\oint_C (x dy - y dx)$, taken counterclockwise.

Suppose now that $\int Pdx+Qdy$ is independent of path in a domain (open connected set) D, while $\partial P/\partial y-\partial Q/\partial x\neq 0$ at a point (x_0, y_0) of D. From continuity there exists a neighborhood N of (x_0, y_0) throughout which $\partial P/\partial y-\partial Q/\partial x$ maintains the same sign. Then, if ΔA is the element of area of a subregion ΔR inside N, the areas of the corresponding projected regions in the XZ- and YZ-planes are approximately equal to $|\cos \beta_P/\cos \gamma_P|\Delta A=|\partial P/\partial y|\Delta A$ and $|\cos \alpha_Q/\cos \gamma_Q|\Delta A=|\partial Q/\partial x|\Delta A$, respectively, where β_P, γ_P and α_Q, γ_Q refer to corresponding direction angles of normals to $z=P(x, y)$ and $z=Q(x, y)$ at the points on each surface above $(x_0\ y_0)$. These approximations follow at once from the approximation $\Delta S\approx|\sec \gamma|\Delta A$ for the surface area above an element ΔA. One of the projections is illustrated in Figure 3, in which $[-\partial P/\partial x, -\partial P/\partial y, 1]$ denotes a set of direction numbers of the normal to $z=P(x, y)$ at

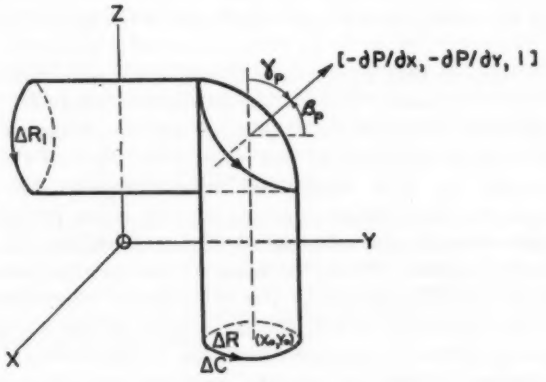

FIG. 3

$(x_0, y_0, P(x_0, y_0))$. Hence, as independence of path in D is clearly equivalent to the line integral being zero around every simple closed path in D, we have from the above discussion that in case $\partial P/\partial y$ and $\partial Q/\partial x$ are each $\neq 0$ at (x_0, y_0), moving counterclockwise,

$$0=\oint_{\Delta C} Pdx+Qdy \approx -\left(\operatorname{sgn}\frac{\partial P}{\partial y}\right)\left|\frac{\partial P}{\partial y}\right|\Delta A+\left(\operatorname{sgn}\frac{\partial Q}{\partial x}\right)\left|\frac{\partial Q}{\partial x}\right|\Delta A \tag{1}$$
$$\approx\left(-\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right)\Delta A$$

with all partials evaluated at (x_0, y_0), for ΔC the boundary of a sufficiently small region ΔR. A relation of type (1) also holds in case $\partial P/\partial y=0$ but $\partial Q/\partial x\neq 0$ at (x_0, y_0). For, if there exists a neighborhood N_1 of (x_0, y_0) throughout which $\partial P/\partial y=0$, choose ΔR inside N_1 and in a neighborhood in which $\partial Q/\partial x$ maintains the same sign. On the other hand, if every neighborhood of (x_0, y_0) contains a

point at which $\partial P/\partial y \neq 0$, then there exists a sequence of points $(x_i, y_i) \to (x_0, y_0)$ at each of which $\partial P/\partial y \neq 0$. Moreover, in a neighborhood ΔR of (x_0, y_0) in which $\partial Q/\partial x$ maintains the same sign, $\partial P/\partial y$ and $\partial Q/\partial x$ cannot be equal at infinitely many of the (x_i, y_i), for continuity would demand their equality at (x_0, y_0). Thus, there exists at least one point $(\bar{x}, \bar{y})$ at which $\partial P/\partial y \neq \partial Q/\partial x$ with each term $\neq 0$, and relation (1) would hold for the partials evaluated at $(\bar{x}, \bar{y})$ and ΔC the boundary of a sufficiently small neighborhood of $(\bar{x}, \bar{y})$. The remaining case $\partial Q/\partial x = 0$ but $\partial P/\partial y \neq 0$ at (x_0, y_0) can be treated in a similar manner. Consequently, in each case there exists a point such that for a sufficiently small neighborhood ΔN of it and all simple closed curves ΔC in ΔN, taken counterclockwise and enclosing an area ΔA, $0 = (1/\Delta A) \oint_{\Delta C} P dx + Q dy \approx (-\partial P/\partial y + \partial Q/\partial x)$ evaluated at the point, and $(-\partial P/\partial y + \partial Q/\partial x)$ maintains the same sign in ΔN. As the approximation can be made as close as desired by choosing a suitably small ΔN about the point, we have a contradiction to the inequality of $\partial P/\partial y$ and $\partial Q/\partial x$ in D.

Conversely, suppose that $P(x, y)$ and $Q(x, y)$ have continuous first partials in a simply connected domain D and that $\partial P/\partial y = \partial Q/\partial x$ in D. From the geometric interpretation above, $\oint_C P dx + Q dy$, for any simple closed curve C in D, is the algebraic sum of the areas of two projections. Now, if the region in the XY-plane bounded by C is subdivided into subregions ΔR with ΔA the area of a typical one, then the areas of the corresponding projections ΔR_1 and ΔR_2 will be approximately given by $|\partial P/\partial y| \Delta A$ and $|\partial Q/\partial x| \Delta A$, respectively, as in the paragraph above. Thus, the areas of the two projected subregions differ at most by an infinitesimal of the third order. Moreover, for a given orientation of the boundary of ΔR, the boundaries of the corresponding ΔR_1 and ΔR_2 will be traversed in opposite directions. Consequently, by considering the counterclockwise line integral around C as the sum of counterclockwise line integrals around the boundaries of the subregions, the interior line integrals will cancel; and, hence, it follows that $\oint_C P dx + Q dy = 0$, guaranteeing independence of path in D.

References

1. W. V. Lovitt, A geometrical interpretation of Green's formula, this MONTHLY, vol. 22, 1915, pp. 152–154.
2. Wilfred Kaplan, Advanced Calculus, Cambridge, 1952.
3. E. J. Townsend, Functions of a Complex Variable, New York, 1915.

CORRECTION

R. D. Larsson, *General solutions of linear ordinary differential equations*, this MONTHLY, vol. 65, 1958, pp. 523–525. In lines 1 and 2 on page 525, $F(x, k_2)$ should be replaced by $cF(x, k_2)$.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

MAA VISITING LECTURESHIP PROGRAM FOR SECONDARY SCHOOLS 1958–59

At the meeting of the Board of Governors in Boston on August 25, 1958, it was announced that the Mathematical Association of America had received a National Science Foundation grant of $47,700 to sponsor a visiting lectureship program for secondary schools as a pilot program for this school year and the school year of 1959–60. The program is being carried on in secondary schools in six regions during the second semester of 1958–59. In general, the plan follows that of the Association's visiting lectureship program for colleges, successfully operated since 1954. The general aims of the visiting lectureship program for secondary schools are listed as the following:

a. To strengthen and stimulate the mathematics programs of secondary schools;
b. To encourage co-operation between college and secondary school mathematics staffs;
c. To provide the mathematics staff and the students in secondary schools with an opportunity for personal contacts with productive and creative mathematicians;
d. To aid in the motivation of secondary school students to consider careers in mathematics and the teaching of mathematics.

It is expected that the visiting lecturers will spend one day in a school, except in California, where most of the visits will be on a half-day basis. In the larger cities the lecturer may plan to stay several days and visit a number of secondary schools, both in the city and in the neighboring smaller communities. It is hoped that teachers from a number of schools may meet the lecturer in a joint meeting and that it also may be possible for students or selected groups of students from several schools to meet together in assembly to hear the lecturer.

The lecturers who have agreed to work with the program during the second semester 1958–59 are as follows:

Richard D. Anderson, Louisiana State University
John D. Baum, Oberlin College
William E. Briggs, University of Colorado
James A. Cooley, University of Tennessee
James Clifton Eaves, University of Kentucky
William H. Fagerstrom, Pan American College
William Thomas Guy, Jr., University of Texas
Cletus O. Oakley, Haverford College
Henry W. Syer, Kent School, Kent, Connecticut
Verne J. Varineau, University of Wyoming

The six regions for 1958–59, and the committee representative of each region, are:

Colorado-Wyoming	B. W. Jones
Indiana	Marie S. Wilcox
Louisiana-Mississippi	Houston T. Karnes
Massachusetts-New Hampshire	W. E. Ferguson
Northern California	Roy Dubisch
Tennessee	F. A. Ficken

Most of the lecturers will work on a part-time rather than a leave-of-absence basis, during the second semester of this year. Dr. Anderson will spend most of his time in Louisiana and Mississippi; Dr. Baum and Dr. Syer have been assigned primarily to Massachusetts and New Hampshire; and Dr. Cooley and Dr. Oakley to Tennessee. Several of the lecturers will work in Indiana and Colorado-Wyoming. In Northern California the program is to be carried out in complete cooperation with the Northern California Section of the Association. The plan of operation there will be similar to that followed by the Northern California Section last year. The lecturers in Northern California will be local people who will travel to schools short distances from their colleges. The California lecturers are not listed above, and there may be a few other lecturers not listed who will be available for short trips in some of the other regions.

To date, there is indication of good cooperation from school people, including state departments of education. In one state at least—Tennessee—the state department of education has mailed the announcements of the program to all high schools.

SOME EDUCATIONAL ACTIVITIES IN PHYSICS

W. C. Kelly, American Institute of Physics

Events of the past year have brought physicists, as well as their colleagues in the other natural sciences and mathematics, increased opportunities to improve the teaching of their subject at all levels of education. The history of these educational efforts in physics runs far back into the pre-Sputnik era, but a brief report on several recent developments may be of interest to mathematicians.

First, a note about the organizations involved in activities reported here is appropriate. The American Institute of Physics (AIP) is a federation of five principal physics societies in the United States: the American Physical Society, the Optical Society of America, the Acoustical Society of America, the Society of Rheology, and the American Association of Physics Teachers (AAPT). At its headquarters in New York City, AIP carries on those activities which can best be done by a central office in support of research and teaching in physics and the furthering of an appreciation of physics among the general public. Publication of journals in physics, public relations, and education and manpower are the three areas in which AIP is active. Its Education Department, established within the last year, includes the writer and secretarial assistants as a permanent core staff plus other professional personnel who join the AIP staff

on leave from physics departments to work on special educational projects. By June of this year, the AIP Education Department will have a full-time staff of eight. The AIP Education Department works closely with all of the Member Societies. Since the AAPT has a special interest in education in physics, however, many of the educational projects described here are joint AAPT-AIP activities, administered by AIP under policies jointly determined by AAPT and AIP.

The Visiting Scientists Program in Physics under a National Science Foundation grant resembles in its objectives the program of visits by mathematicians to colleges and need not be discussed in detail here. However, the physics program relies upon a rather large number of visitors—about eighty this year—each making only one or two visits to colleges. Another part of the program of visits by physicists provides for visits to high schools to assist teachers and stimulate interest in physics. About three hundred visiting days will be devoted to secondary schools this year.

The Project on the Design of Physics Buildings was launched this year under a grant from the Educational Facilities Laboratories. A preliminary survey of the building plans of physics departments in colleges and universities revealed that almost a quarter of a billion dollars will be spent in the next decade by educational institutions to meet increasing enrollments and expanding research programs in physics. This project will provide information for physicists, architects, and administrators about the best design features of present physics building facilities in schools and colleges as well as estimates of how the changing space requirements in physics can be met in the future. Professor R. Ronald Palmer of Beloit College will direct this project. The project will also have the services of a full-time architect and of consultants. Since science buildings on college campuses are often shared by several academic departments, the project staff will seek the advice of mathematicians and scientists in other areas as the work proceeds. That the design of physics building facilities is an educational problem can be attested by anyone who has struggled with a poorly-designed lecture room or teaching laboratory.

Equipment for illustrating the principles of physics is important to physics teachers at all levels of education. At the secondary school level, AAPT and AIP are seeking to provide assistance to school districts which wish to improve their stock of physics teaching equipment. Advice by physicists during their visits to schools is but one of the ways being explored. At the college level, AAPT and AIP are conducting an apparatus drawings project in which complete shop drawings of new apparatus developed in physics departmental shops will be made available to other physics departments for local construction. Further to stimulate interest in new apparatus, an AAPT apparatus competition was held in New York on January 29–31, 1959. Prizes were awarded on items of equipment which were judged most likely to advance the teaching of introductory college physics.

Physicists are greatly interested in recent developments in the teaching of

college mathematics, both because these developments are interesting in their own right and because they affect the teaching of college physics. At present there is a great need for physicists to become better informed about these developments and to discuss them with their colleagues in mathematics. AAPT and the Mathematical Association of America are in the process of appointing a joint committee to stimulate discussion of the new mathematics program by physicists and mathematicians by means of an exchange of journal articles and by colloquia at national meetings.

Numerous other educational projects in physics are well advanced: publication of a booklet to stimulate interest among students in physics and mathematics courses in high schools, publication of a booklet for the improvement of the high school physics course, evaluation of the effectiveness of the teaching of physics by television and film, stimulation of research in liberal arts colleges, revision of the introductory college physics course, a review of the physics content of elementary school science, awards to secondary schools outstanding for their work in science, physics tests, and a book of advanced laboratory experiments.

These are some of the educational projects being conducted by AAPT and AIP. In addition, physicists are interested in other experimental approaches to the improved teaching of physics, which are described elsewhere, such as the physics course for high schools being developed by the Physical Science Study Committee, the physics films made by Professor Harvey White for high schools, and the Continental Classroom "Atomic Age Physics" which Professor White is conducting on television.

TEXAS COMMISSION TO STUDY MATHEMATICS

JOHN WAGNER, The University of Texas

On November 14, 1955, the State Board of Education for the State of Texas authorized a study of graduation requirements in accredited high schools. This study was in recognition of the widespread interest in Texas in the aims and accomplishments of public schools.

In addition to recommendations on graduation requirements the Board and a special Advisory Committee made a far-reaching recommendation. The suggestion was that a detailed study be made of the approved courses which are offered in Texas schools and that the State Board specify the objectives and general content of each course.

A Commission to Study the Mathematics Curriculum in Texas Elementary and Secondary Schools has been appointed and should have its studies completed during the calendar year of 1959.

The original membership of the Mathematics Study Commission was composed as follows: six public school teachers (two elementary, two junior high and two senior high), three college teachers (two of whom shall be instructors of pure and applied mathematics—Dr. D. E. Edmondson, Southern Methodist

University and Dr. W. T. Guy, Jr., The University of Texas—and one of whom shall be an instructor in mathematics education—Dr. Joyce Benbrook, University of Houston), three public school principals (one elementary, one junior high, and one senior high), two curriculum directors or supervisors, and two school board members. Recently the State Board of Education appointed three more members, a high school teacher, a supervisor, and a college instructor of mathematics education, Mr. Don Cude, Southwest State College.

Consultants to the Commission have included Mr. Ralph Stafford, Texas Education Agency, and Mr. John Wagner, Extension Division of The University of Texas.

The purposes of the Study include the following:

1. To review the content of the present mathematics curriculum and to consider the changes in content which have taken place in recent years in order to meet present-day needs for all children.
2. To study experimental programs for accelerated students of high mathematics ability and to consider recommendations which will allow these students to be given advanced standing in mathematics.
3. To consider a program which will meet the mathematical needs of pupils who are not college bound.
4. To consider successful practices now being used for the early indentification and proper development of students with high aptitude in mathematics and to recommend ways for encouraging greater use of these practices.
5. To assist the State Department of Education in the preparation of curriculum materials which will give direction to those who wish to teach the combined Plane and Solid Geometry course, the Advanced Mathematics course, and other courses in secondary mathematics in which a change in emphasis is taking place.
6. To seek the cooperation of business and industry in providing for students enriched mathematics experiences both in school and out of school.

SCIENTIFIC MANPOWER BULLETIN

Scientific Manpower Bulletin of the National Science Foundation, Number 9, October, 1958, is devoted to a report on foreign-language knowledge of American scientists, 1954–55. Over 97,000 scientists in the National Register of Scientific and Technical Personnel reported a knowledge, with varying degrees of proficiency, of at least one foreign language. This is about three-quarters of the total in the Register 1954–55. German was the language most often reported. The fields of chemistry and chemical engineering contained the largest numbers reporting a language competency. Among mathematicians, 438 indicated some proficiency in the Balto-Slavic languages and 104 reported some proficiency in the Tibeto-Chinese languages.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1361. *Proposed by M. S. Klamkin, AVCO Research and Development*

If A, B, C are angles of a triangle, show that

$$\csc A/2 + \csc B/2 + \csc C/2 \geqq 6.$$

E 1362. *Proposed by Marlow Sholander, Carnegie Institute of Technology*

Consider a vertical girl whose waist is circular, not smooth, and temporarily at rest. Around the waist rotates a hula hoop of twice its diameter. Show that, after one revolution of the hoop, the point originally in contact with the girl has traveled a distance equal to the perimeter of a square circumscribing the girl's waist.

E 1363. *Proposed by Leo Moser, University of Alberta*

Let numbers be written to base b where b has the form $b=r^2+1$. Given r consecutive numbers, the last divisible by r, then the digital root of their sum is $1+2+\cdots+r=r(r+1)/2$.

E 1364. *Proposed by James Serrin, University of Minnesota*

In a plane, let A denote a closed convex curve in contact with a given curve C. Also, let B denote the mirror image of A across the tangent line to C at the point of contact. Suppose that the curvatures of A, B, C permit A and B to roll without slipping along their respective sides of C. Then, as A rolls along C, let A' denote the roulette traced out by the point of A initially in contact with C. Similarly, let B' denote the roulette generated by rolling B on C. Show that the area enclosed by an arch of A' and the corresponding arch of B' is independent of C. In particular, if A is a circle the enclosed area is just six times the area of A.

E 1365. *Proposed by Melvin Hausner, Stevens Institute of Technology*

Let H be the class of polynomials $f(x)$ with rational coefficients such that $f(n)$ is an integer when n is an integer. Prove that a sequence a_n of $+1$'s and -1's is of the form $a_n=(-1)^{f(n)}$, $f(x)\in H$, if and only if it is a periodic sequence of period 2^k.

SOLUTIONS

An Inequality in Two Variables

E 1331 [1958, 627]. *Proposed by P. L. Duren, Massachusetts Institute of Technology*

Show that, for all $a \geqq 0$ and $b \geqq 1$,

$$ab \leqq e^a + b(\ln b - 1),$$

with equality if and only if $b = e^a$.

Solution by H. D. Lipsich, University of Cincinnati. The inequality is true under the less restrictive conditions, a arbitrary, $b > 0$, as follows. It is well known that $e^u \geqq 1 + u$ for all u, with equality holding iff $u = 0$. Thus for arbitrary x and y, putting $u = x - y$ and rearranging, we have

$$xe^y \leqq e^x + e^y(y - 1),$$

from which the desired result follows on putting $x = a$ and $y = \ln b$.

It is of some interest to see a solution with which the restrictions placed by the proposer are naturally associated. To this end suppose $a \geqq 0$, $b \geqq 1$, and consider the curve $y = e^x$. A glance at the graph shows that the area of the rectangle with sides a and b is less than or equal to the area between the curve and the x-axis from 0 to a, plus the area between the curve and the y-axis from 1 to b. That is,

$$ab \leqq \int_0^a e^t dt + \int_1^b \ln t dt,$$

which, upon integration, yields the desired inequality. It is geometrically clear that equality holds iff the point (a, b) is on the curve.

Also solved by Derry Breault, D. R. Brillinger, J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), Ian Connell, A. E. Danese, Underwood Dudley, David Eakin, E. L. Ellis, P. G. Engstrom, Michael Goldberg, A. G. Grace, Jr., Bernard Greenspan, Cornelius Groenewoud, P. G. Hodge, Jr., Richard Holt, D. I. Knee and G. M. Leibowitz and N. Metas and L. N. Patterson and G. R. Stoodley (jointly), Morton Kupperman, Joe Lipman, Richard McChesney, D. C. B. Marsh, J. W. Mettler, G. J. Michaelides, C. E. Miller, Joseph Muskat, C. S. Ogilvy, F. R. Olson, W. V. Parker, C. L. Perry, Stanton Philipp, C. F. Pinzka, S. C. Port, K. H. Pyle, L. A. Ringenberg, D. A. Robinson, Jack Roseman, Vencil Skarda, J. A. Tierney, R. J. Wagner, Chih-yi Wang, Dale Woods, W. A. Veech, Julius Vogel, David Zeitlin, and the proposer. Late solutions by Robert Bart, S. H. Greene, Norman Greenspan, Vern Hoggatt, A. R. Hyde, J. D. E. Konhauser, Helen M. Marston, M. D. Mavinkurve, and J. L. Pietenpol.

Two Restricted Trigonometric Identities

E 1332 [1958, 627]. *Proposed by P. L. Chessin, University of Maryland*

If A, B, C are the angles of a triangle and x is such that

$$\cos(x + A)\cos(x + B)\cos(x + C) + \cos^3 x = 0,$$

then

$$\tan x = \cot A + \cot B + \cot C, \tag{1}$$

$$\sec^2 x = \csc^2 A + \csc^2 B + \csc^2 C. \tag{2}$$

Solution by David Zeitlin, Remington Rand Univac. Using

$$\cos (u + v) = \cos u \cos v - \sin u \sin v,$$

the equation simplifies to $\sec^2 x(\tan x - m) = 0$, where

$$m = \cot A + \cot B + \cot C.$$

It follows that $\tan x = m$, which is (1).

For (2) we note that

$$\begin{aligned} \sec^2 x &= 1 + \tan^2 x \\ &= \cot^2 A + \cot^2 B + \cot^2 C + 3 \\ &= (1 + \cot^2 A) + (1 + \cot^2 B) + (1 + \cot^2 C) \\ &= \csc^2 A + \csc^2 B + \csc^2 C. \end{aligned}$$

Also solved by A. N. Aheart, Walter Bloch, D. R. Brillinger, Leonard Carlitz, J. W. Clawson, E. L. Ellis, Mildred Going and Bert Levy (jointly), Cornelius Groenewoud, J. R. Hendricks, P. G. Hodge, Jr., Sidney Kravitz, Joe Lipman, D. C. B. Marsh, G. J. Michaelides, Stewart Nagler, T. L. Reynolds, E. R. Vance, Chih-yi Wang, W. V. Webb, Carole Weiss, Charles Wexler, R. H. Wilson, Jr., and the proposer. Late solutions by Robert Bart, D. A. Breault, A. E. Danese, S. H. Greene, Louise Grinstein, E. L. Hubbard, J. B. Muskat, and Roscoe Woods. Woods pointed out that x is the complement of the Brocard angle of triangle ABC.

A Special Determinant

E 1333 (corrected) [1958, 627]. *Proposed by V. F. Ivanoff, San Carlos, California*

If $s_k = 1^k + 2^k + \cdots + n^k$, show that

$$\begin{vmatrix} s_1 & 1 & 0 & 0 & \cdot & 0 \\ s_2 & s_1 & 2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_{n-2} & \cdot & \cdot & \cdot & n-1 \\ s_n & s_{n-1} & \cdot & \cdot & \cdot & s_1 \end{vmatrix} = (n!)^2.$$

Solution by Leonard Carlitz, Duke University. The following more general result may be of interest. Let $x_1, \cdots, x_n$ be arbitrary numbers, let

$$s_k = x_1^k + x_2^k + \cdots + x_n^k,$$

let D denote the given determinant under the more general interpretation, and let $p_1, \cdots, p_n$ denote the elementary symmetric functions of the x's. Then it is familiar that

$$p_0 s_k - p_1 s_{k-1} + p_2 s_{k-2} - \cdots + (-1)^{k-1} p_{k-1} s_1 + (-1)^k k p_k = 0$$

for $k=1, \cdots, n$ and where $p_0=1$. We may consider this a system of equations in the unknowns $p_0, -p_1, \cdots, (-1)^{n-1} p_{n-1}$. Solving for p_0 we get $p_0 = n! p_n / D$, or $D = n! p_n$.

Also solved by D. R. Brillinger, E. A. Fay, J. H. Hodges, Joe Lipman, W. R. McEwen, D. C. B. Marsh, W. V. Parker, B. M. Stewart, and the proposer.

Property of a Simply Periodic Entire Function

E 1334 [1958, 628]. *Proposed by Burton Randol, The Rice Institute*

If $f(z)$ is any simply periodic entire function, show that there exists a (finite) z_0 such that $f(z_0) = z_0$.

Solution by Arthur Rosenthal, Purdue University. Let f be nonconstant and let $h \neq 0$ be a period of f. By Picard's theorem, the entire function $f(z) - z$ assumes either the value 0 or otherwise the value h at a certain z_1. In the former case we have $f(z_1) = z_1$; in the latter case we have $f(z_1 + h) = f(z_1) = z_1 + h$. We may therefore take $z_0 = z_1$ or $z_0 = z_1 + h$.

Also solved by Eugene Albert, J. L. Brown, Jr., Ian Connell, D. S. Greenstein, Joe Lipman, Joseph Muskat, Stanton Philipp, W. F. Trench, and the proposer.

Fixed Point of a Contraction Mapping

E 1335 [1958, 628]. *Proposed by C. N. Campopiano, Polytechnic Institute of Brooklyn*

Let $f(z)$ be a complex function of the complex variable z defined for all finite z. Suppose there is a constant k, $0 < k < 1$, such that

$$|f(z) - f(w)| \leqq k |z - w|$$

for all z, w. Show that there is a unique solution to the equation

$$z = f(z) + a,$$

where a is an arbitrary constant.

I. *Solution by Peter Treuenfels, Brookhaven National Laboratory.* Choose z_0 arbitrarily, and recursively define

$$z_{n+1} = f(z_n) + a. \tag{1}$$

It follows from the hypothesis, by induction on j, that

$$|z_{j+1} - z_j| \leqq k^j |z_1 - z_0|.$$

Since $0 < k < 1$, the series

$$\sum_{j=0}^{\infty} (z_{j+1} - z_j)$$

converges absolutely. Therefore

$$z_{n+1} = z_0 + \sum_{j=0}^{n} (z_{j+1} - z_j)$$

tends to a limit, say z. The hypothesis on f implies that f is continuous. Hence, passing to the limit in (1), we see that

$$z = f(z) + a. \tag{2}$$

Thus a solution exists. If there were two solutions z' and z'', then, by subtraction, (2) would imply that $z'-z''=f(z')-f(z'')$. But this is compatible with the hypothesis on f only if $z'=z''$.

See (a) L. Collatz, *Einige Anwendungen funktionanalytischer Methoden in der praktischen Analysis*, Z. Angew. Math. Phys., vol. 4, 1953, pp. 327–357, esp. Sec. 2; (b) L. Bers, *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*, New York, 1958, p. 57.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* For $k<1$ there exist finite values of z, say z_0, such that

$$|f(0) + a| /(1 - k) = |z_0| .$$

Then one has successively,

$$\begin{aligned} |f(z_0) + a| &\leqq |f(z_0) - f(0)| + |f(0) + a| \\ &\leqq k|z_0| + |f(0) + a| \text{ (by hypothesis of problem)} \\ &\leqq |z_0| \text{ (by definition of } z_0). \end{aligned}$$

Thus the region R, $|z| \leqq |z_0|$, will be mapped continuously by $z \to f(z)+a$ into itself. Therefore, by Brouwer's fixed point theorem, there exists at least one Z in R such that $Z=f(Z)+a$.

The existence of two solutions, Z and Z', would imply that $|f(Z)-f(Z')| = |Z-Z'|$, contrary to the hypothesis that $|f(z)-f(w)| < |z-w|$.

Also solved by D. R. Brillinger, J. L. Brown, Jr., Joseph Geiser, D. A. Kearns, Gerald Leibowitz, Joe Lipman, C. S. Ogilvy, Stanton Philipp, W. F. Trench, W. A. Veech, David Zeitlin, and the proposer. Late solutions by C. H. Cunkle and D. A. Freedman.

Geiser found the problem, with solution, in Kolmogoroff, *Elements of the Theory of Functions and Functional Analysis*, vol. 1.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4765 [1957, 746], Corrected. *Proposed by J. L. Massera, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

Let $y=f(x)$ be a real function defined for $x \geqq 0$. If (i) f has a finite upper bound in any finite interval, and (ii) there are two positive numbers h, k such that $x'-x'' \geqq h$ implies $f(x')-f(x'') \geqq k$, then, given $\alpha>1$, there is an increasing function $g(x)$ having as many continuous derivatives as we please, such that $g(x-\alpha h)<f(x)<g(x)$ for all $x \geqq \alpha h$.

4840. *Proposed by P. T. Bateman, University of Illinois*

Suppose $a_0>0$, $a_1 \geqq 0$, $a_2 \geqq 0$, $\cdots$, and put $A_n=a_0+a_1+\cdots+a_n$. Show that if, as $n \to \infty$, $A_n \to +\infty$ and $a_n/A_n \to 0$, then $\sum a_n x^n$ has radius of convergence unity.

4841. *Proposed by John Lamperti, California Institute of Technology*

Let N be a random variable whose values are nonnegative integers. Independently of each other, each of N balls is placed either in urn A with probability p $(0<p<1)$ or in urn B with probability $1-p$, resulting in N_A balls in urn A and $N_B=N-N_A$ in urn B. Show that the random variables N_A and N_B are independent if and only if N has a Poisson distribution.

4842. *Proposed by John Lamperti, California Institute of Technology*

Let X be a nonnegative random variable, and let Y be uniformly distributed on the interval $(0, X)$. Let $Z=X-Y$. Show that Y and Z are independent if and only if X has the density

$$f(x) = \begin{cases} a^2 x e^{-ax}, & x \geqq 0 \\ 0, & x < 0 \end{cases}, \qquad \text{for } a \geqq 0.$$

4843. *Proposed by Leopold Flatto, Brooklyn Polytechnic Institute*

Let S be a set everywhere dense in the plane. Does it follow that S is dense on some line segment in the plane?

4844. *Proposed by D. S. Kahn, Princeton University*

Consider the ring Q of polynomials in the partial differential operators $\partial/\partial x_1, \cdots, \partial/\partial x_n$ over the field of rational functions of $x_1, \cdots, x_n$. (1) Determine all (two-sided) ideals of Q. (2) Prove that Q is a primitive ring.

SOLUTIONS

Postulates for a Ring

4784 [1958, 289]. *Proposed by D. W. Jonah, Purdue University*

In John L. Kelley, *General Topology*, p. 18, a definition of a ring is given in which the left and right distributive laws are replaced by the composite distributive law:

$$(u + v)(x + y) = ux + uy + vx + vy.$$

(a) Show by an example that such a system is not necessarily a ring. (b) Show that if such a system contains an element a such that $a0=0$ (in particular, if the system has a multiplicative identity) then the system is a ring.

Solution by R. A. Beaumont, University of Washington, Seattle. (a) Let $\{u\}$ be an additive cyclic group of order 3, and define the product of every pair of elements to be u. Then $(u+v)(x+y)=u$ and $ux+uy+vx+vy=u$, so that the composite distributive law is satisfied. Multiplication is associative and commutative, but the system is not a ring since $00=u\neq 0$.

(b) For all b,

$$b0 = (b + 0)(0 + 0) = b0 + b0 + 00 + 00,$$
$$0b = (0 + 0)(b + 0) = 0b + 00 + 0b + 00,$$

so that $b0=0b=-00-00$ and, in particular, $a0=-00-00$. Therefore $a0=0$ implies $b0=0b=0$ for all b. Thus we have

$$(a + b)c = (a + b)(c + 0) = ac + a0 + bc + b0 = ac + bc,$$
$$a(b + c) = (a + 0)(b + c) = ab + ac + 0b + 0c = ab + ac,$$

so that the system is a ring.

Also solved by Auguste Forge, Fred Galvin, E. R. Gentile, W. G. Leavitt, Joe Lipman, F. D. Parker, G. W. Potnick, D. A. Robinson, Azriel Rosenfeld, Tôru Saitô, J. E. Simpson, Marlow Sholander, Paul Slepian, Robert Spira, M. J. Walsh, Thann Ward, C. R. B. Wright, and the proposer.

Editorial Note. An interesting generalization is given by Saitô. See *Note on the distributive laws* in this issue of the MONTHLY, pp. 280–283.

Vectors in Binary n-space

4794 [1958, 451]. *Proposed by S. W. Golomb, California Institute of Technology*

Let V_n be binary n-space (the collection of n-vectors over the field of two elements). Consider two vectors of V_n to be in the same class if they differ only

by a cyclic permutation of their components. Show that the number of classes is even, except when $n=2$.

Solution by John B. Kelly, Pennsylvania State University. Call two vectors of V_n conjugate if one is obtained from the other by replacing its 0 components by 1 and its 1 components by 0. With each class in V_n we may associate the class of conjugate vectors. Thus, if there are no self-conjugate classes, the number of classes is even. This is clearly the case when n is odd, for the vectors in a self-conjugate class must have the same number of 1 and 0 components, so that the total number of components is even.

Let us assume $n=2^m n'$, where n' is odd. We proceed by induction on m. The theorem is true when $m=0$ by what we have just said.

Let us denote by S_k the number of classes in V_n with k elements. Clearly, if $S_k \neq 0$, then k divides n. Since V_n has 2^n vectors, we have

$$2^n = \sum_{k|n} kS_k = \sum_{k|n/2} kS_k + \sum_{2^m|k} kS_k = \sum_{k|n/2} kS_k + \sum_{d|n'} 2^m dS_{2^m d}. \tag{1}$$

If $k|n/2$, then a vector belonging to a class with k elements in V_n can be obtained only by iterating a vector in $V_{n/2}$. The first summand in (1) is thus equal to the number of vectors in $V_{n/2}$, namely $2^{n/2}$. Hence (1) becomes

$$2^n = 2^{n/2} + 2^m \sum_{d|n'} dS_{2^m d}.$$

This gives

$$\sum_{d|n'} dS_{2^m d} = 2^{(2^{m-1}n'-m)}(2^{n/2} - 1).$$

Now $2^{m-1}n' - m > 0$ unless $m=1$, 2, and $n'=1$. Thus, with these exceptions

$$\sum_{d|n'} dS_{2^m d} \equiv 0 \bmod 2.$$

Since n' is odd, all divisors of n' are odd and

$$\sum_{d|n'} S_{2^m d} \equiv \sum_{d|n'} dS_{2^m d} \equiv 0 \pmod 2.$$

Thus the total number of classes in V_n with k elements, where $2^m|k$, is even. The total number of classes in V_n with k elements, where $2^m \nmid k$, is the same as the total number of classes in $V_{n/2}$ which is even by our induction hypothesis, save when $n=4$. This corresponds to an exceptional case above. The other exceptional case corresponds to $n=2$, for which the theorem is false. The theorem is readily verified for $n=4$ by explicit construction of the classes of V_4.

Also solved by N. J. Fine, J. H. Hodges and D. Orloff, and the proposer.

Zeros of a Special Sum

4795 [1958, 451]. *Proposed by Y. L. Luke, Midwest Research Institute, Kansas City, Missouri*

Find the zeros of

$$H_n(x) = \sum_{m=0}^{n} \frac{(-2)^m(2n-m)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m+1)}.$$

Solution by W. A. Al-Salam and L. Carlitz, Duke University. We first show that

$$(*) \qquad H_{n+1}(x) = -2(x-2n-1)H_n(x).$$

Indeed

$$(x-2n-1)H_n(x) = \sum_{m=0}^{n} \frac{(-2)^m(2n-m)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m+1)}((x-m)-(2n-m+1))$$

$$= \sum_{m=0}^{n} \frac{(-2)^m(2n-m)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m)} - \sum_{m=0}^{n} \frac{(-2)^m(2n-m+1)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m+1)}$$

$$= \sum_{m=1}^{n+1} \frac{(-2)^{m-1}(2n-m+1)!\Gamma(x+1)}{(m-1)!(n-m+1)!\Gamma(x-m+1)} - \sum_{m=0}^{n} \frac{(-2)^m(2n-m+1)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m+1)}$$

$$= -\frac{1}{2}\sum_{m=0}^{n+1} \frac{(-2)^m(2n-m+2)!\Gamma(x+1)}{m!(n-m+1)!\Gamma(x-m+1)} = -\frac{1}{2}H_{n+1}(x).$$

This proves (*).

It is now an immediate sequence of (*) and the fact that $H_0(x)=1$ that

$$H_n(x) = (-2)^n \prod_{m=1}^{n} (x-2m+1),$$

whence the zeros of $H_n(x)$ are $1, 3, 5, \cdots, 2n-1$.

Also solved by Robert Breusch, N. J. Fine, D. C. B. Marsh, and the proposer.

"Preferred Numbers" and a Set of Polynomials

4796 [1958, 451]. *Proposed by Chandler Davis, American Mathematical Society, Providence, R. I.*

Say t_1 is preferred to t_2 provided (i) $0 \leqq t_1 < t_2 \leqq 1$, or (ii) $-1 \leqq t_2 < t_1 < 0$, or (iii) $-1 \leqq t_2 < 0 \leqq t_1 \leqq 1$. Determine a sequence of polynomials P_n such that, whenever t_1 is preferred to t_2, $P_n(t_2) = o(P_n(t_1))$.

Solution by the proposer. Define first

$$R_n(t) = -\int_{-1}^{t} t(1-t)^{n-1}dt \qquad (n = 2, 3, \cdots),$$

real polynomials of degree $n+1$. They satisfy

$$R_n'(t)\begin{cases}>0 & (t<0)\\ <0 & (0<t<1),\end{cases} \tag{1}$$

$$R_n(1) = R_n(-1/n), \tag{2}$$

$$R_n(t) > 0 \qquad (-1 < t \leqq 1). \tag{3}$$

Now define $P_n(t)=(R_n(t))^{n!}$, real polynomials of degree $(n+1)!$. They, too, satisfy (1), (2), (3).

Let t_1 be preferred to t_2. If (iii) holds, then take $N>-1/t_2$; for $n>N$, $0<R_n(t_2)<R_N(t_2)<R_N(-1/N)=R_N(1)<R_n(1)\leqq R_n(t_1)$; whence

$$\left\{\frac{R_n(t_2)}{R_n(t_1)}\right\}^{n!} \leqq \left\{\frac{R_N(t_2)}{R_N(t_1)}\right\}^{n!} = o(1).$$

It is therefore enough to consider (i) and (ii). (Also, for convenience in the next paragraph, exclude $t_2=1$; no generality is thereby lost because, in case $0\leqq t_1<t_2'<t_2=1$, $0<P_n(1)<P_n(t_2')<P_n(t_1)$.)

Now

$$\log\frac{P_n(t_2)}{P_n(t_1)} = \log\left\{R_n(t_2)\right\}^{n!} - \log\left\{R_n(t_1)\right\}^{n!} = n!(t_2-t_1)\frac{R_n'(T_n)}{R_n(T_n)}$$

for some T_n between t_1 and t_2. This application of the law of the mean is justified because R_n has no zeros in the interval. One substitutes now in the right-hand side the explicit expressions for R_n' and R_n; then, for either $-1\leqq t_2\leqq T_n\leqq t_1<0$ or $0\leqq t_1\leqq T_n\leqq t_2<1$, a straightforward estimate shows that $\log(P_n(t_2)/P_n(t_1))$ approaches $-\infty$ as $n\to\infty$. This completes the proof.

Remark. Actually, $P_n(t_2)=o(P_n(t_1))$ uniformly in t_2, for t_1 preferred to t_2 and t_2 bounded away from t_1. Any sequence of polynomials having this property provides a solution of a problem (the "induced degeneracy difficulty") left open by the proposer in *Estimating eigenvalues*, Proc. Amer. Math. Soc., vol. 3, 1952, pp. 942–947.

Also solved by Jeny Browkin.

Linear Independence

4797 [1958, 452]. *Proposed by D. J. Newman, Massachusetts Institute of Technology*

Prove that all expressions like $7\sqrt{19}/4-3\sqrt{7}+8\sqrt{6}/5$ are irrational. More specifically, prove that the square roots of the square-free integers are linearly independent over the rationals.

Solution by W. F. Furr and L. K. Williams, Southern University, Baton Rouge, La.

If p is any prime then $\sqrt{p}$ is irrational. Assume that r is the smallest integer such that there are r primes $p_1, \cdots, p_r$ and k square-free integers $M_1, \cdots, M_k$ which are products of these primes such that

$$a_1\sqrt{M_1} + \cdots + a_k\sqrt{M_k} = R, \tag{1}$$

where R is rational and the a_i are rational and not all zero. (The M_i are power-products of the p_i with all exponents 1.)

Let $M_{i_1}, \cdots, M_{i_s}$ be the M's which have p_r as a factor and write (1) as

$$a_{i_1}\sqrt{M_{i_1}} + \cdots + a_{i_s}\sqrt{M_{i_s}} = R - a_{j_1}\sqrt{M_{j_1}} - \cdots - a_{j_t}\sqrt{M_{j_t}} \tag{2}$$

where we may take $a_{i_1} \neq 0, \cdots, a_{i_s} \neq 0$.

Now square both sides of (2) and note that we can write the result, after transposing, in the form

$$c_1\sqrt{M_1'} + \cdots + c_k\sqrt{M_k'} = C, \tag{3}$$

where C and the c_i are rational and the c_i are not all zero and, further, the M_i' are power-products of the $r-1$ primes $p_1, \cdots, p_{r-1}$. The contradiction proves the theorem.

Also solved by R. P. Langlands, P. E. Schweitzer and H. F. Mattson, and the proposer.

Generalization of a Theorem of Liouville

4798 [1958, 529]. *Proposed by John McCarthy, Dartmouth College*

Let $\alpha_1, \cdots, \alpha_n$ be algebraic numbers linearly independent over the rationals. Show that there is a positive constant C and an integer N such that if $m_1, \cdots, m_n$ are rational integers not all zero then

$$|m_1\alpha_1 + \cdots + m_n\alpha_n| \geqq \frac{C}{\{|m_1| + \cdots + |m_n|\}^N}.$$

This generalizes Liouville's theorem on the approximation of algebraic numbers by rationals.

Solution by the proposer. Let Q be a positive integer such that $Q\alpha_1, \cdots, Q\alpha_n$ are all algebraic integers, *i.e.*, satisfy algebraic equations with integer coefficients with leading coefficient one. Let $N+1$ be the degree of a normal extension of the field of rationals containing all the α_i, and let G be the Galois group of the extension. Since the sum of algebraic integers is an algebraic integer $Q(m_1\alpha_1 + \cdots + m_n\alpha_n)$ is an algebraic integer and the product

$$Q^{n+1} \prod_{g \in G} g(m_1\alpha_1 + \cdots + m_n\alpha_n)$$

will be a rational integer not zero.

Hence we have

$$|m_1\alpha_1 + \cdots + m_n\alpha_n| \geqq \frac{1}{Q^{N+1}\prod_{g\neq 1}|g(m_1\alpha_1 + \cdots + m_n\alpha_n)|}.$$

Letting A be the maximum of the absolute values of the conjugates of the α_i and taking $C=1/Q^{N+1}A^N$ we get the stated inequality.

Solved also by O. Buchta.

Sequence and Subsequence

4800 [1958, 530]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $u(n)$ be a sequence with a recurrence relation of the form

$$u_{n+1} = a_0u_n + a_1u_{n-1} + \cdots + a_ku_{n-k}$$

where $a_0, \cdots, a_k$ are real constants. Show that the sequence obtained by taking every rth term of this sequence satisfies a recurrence of the same length; *i.e.*, if $v_n=u_{rn}$ for $n=0, 1, \cdots k$ there exist constants $A_0, \cdots, A_k$ such that for $n=k+1, k+2, \cdots$,

$$v_{n+1} = A_0v_n + A_1v_{n-1} + \cdots + A_kv_{n-k}.$$

Solution by A. C. Aitken, University of Edinburgh, Scotland. The characteristic equation of the recurrence relation for $u(n)$ is

$$\lambda^k - a_0\lambda^{k-1} - a_1\lambda^{k-2} - \cdots - a_k = 0,$$

and if its roots, λ_j, are distinct, then $u_t=\sum c_j\lambda_j^t$. Hence $v_t=\sum c_j\lambda_j^{rt}$ so that v_t, like u_t, obeys a linear difference equation of order k, the roots of the characteristic equation being λ_j^r.

The case of multiple roots introduces no difficulty.

Also solved by O. Buchta, L. Carlitz, R. C. Lyness, Imanuel Marx, Paul Payette, W. F. Trench and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R.V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

COLLEGE TEXTBOOKS

Editorial Note. Every known American publisher of mathematical books was asked to list the mathematical books published during the five-year period 1954–58. The following list was compiled by checking this list against the MONTHLY files and other known book

lists. Every effort has been made to prepare as complete a list as possible. Prices, authors, *etc.* have been triple checked. We fear, however, that there may still be omissions or errors, and apologize in advance for these. If your book was published between 1954–58, but is not listed, complain to your publisher about it!

Titles have been grouped into the following categories: Mathematics of Finance and Business Mathematics; Remedial (Intermediate) Algebra; Trigonometry; College Algebra; College Algebra and Trigonometry; Analytic Geometry; Integrated Freshman Mathematics; Calculus with Analytic Geometry; Calculus; Post-Calculus Engineering Mathematics; Advanced Calculus; Differential Equations; Partial Differential Equations; Series; Applied Mathematics; Complex Variables; Real Variables; Vector Analysis; Modern Abstract Algebra; Theory of Numbers; Numerical Analysis; Computers; Tables; Geometry; Statistics and Related Topics; Probability; Topology and Algebraic Geometry; Logic and Set Theory; Teaching; Unclassified.

In each case the publisher or the author was asked to stipulate into which category his book was placed, and sent a complete mimeographed listing for final checking. Within each category books are classified by age and, within any given year, alphabetically by author. In each case we have tried to give the title, author(s), his institution, publisher, copyright date, number of pages, current price, and the page and year of the MONTHLY in which it was reviewed. A few titles are followed by a brief comment. These do not signify special merit or demerit, but indicate the content when the title is not sufficiently descriptive, or the book is not well known or has not been reviewed in the MONTHLY.

Publishers will usually send an examination copy (which may be kept or returned for full credit) upon request from any school, college, or university mathematics department. We urge you to examine suitable books for yourself. Only you can decide which book best fits your situation.

MATHEMATICS OF FINANCE AND BUSINESS MATHEMATICS

Mathematics of Business Affairs. By Feldman (Beaumont High School, University City, Mo.). Allyn and Bacon, 1958. 244 pages (workbook format), $3.95. To be reviewed.

Mathematics of Investment, Fourth Edition. By Hart (University of Minnesota). D. C. Heath, 1958. 343 pages plus 150 tables, $6.75. Review: p. 377, 1958.

Mathematics in Business. By Lowenstein (Arizona State College). Wiley, 1958. 364 pages, $4.95. To be reviewed.

Business Mathematics, Fourth Edition. By Richtmeyer and Foust (Central Michigan College). McGraw-Hill, 1958. 412 pages, $5.75. To be reviewed.

A revision of a 1936 text.

Essential Business Mathematics, Third Edition. By Snyder (San Francisco City College). McGraw-Hill, 1958. 470 pages, $5.50. To be reviewed.

Fundamentals of Business Mathematics. By Knick (Marquette University). Richard D. Irwin, 1957. 452 pages, $6.50 text, $7.80 trade.

Mathematics of Finance. By Lee (University of No. Carolina). Richard D. Irwin, 1957. 344 pages, $6.50 text, $7.80 trade.

The Mathematics of Investment. By Osborn (University of Texas). Harper, 1957. 162 pages plus 117 pages of tables, $4.25. Review: p. 683, 1957.

Mathematics of Finance. By Stelson (Michigan State University). Van Nostrand, 1957. 327 pages, $5.50. To be reviewed.

The Mathematics of Finance. By Cissell and Cissell (Xavier University). Houghton Mifflin, 1956. 198 pages, $4.50. Review: pp. 206–7, 1957.

Mathematics of Finance, Second Edition. By Hummel and Seebeck (University of Alabama). McGraw-Hill, 1956. 372 pages, $5.00.

A revision of Hummel and Seebeck's 1948 text.

Mathematics of Finance. By Parker (Texas Technological College). Prentice-Hall, 1956. 288 pages, $5.75.

Elements of Business Mathematics for College Students. By Snyder (San Francisco City College). McGraw-Hill, 1956. 249 pages, $3.90.

Mathematics of Business, Accounting, and Finance. By Trefftzs (University of Southern California) and Hills (Los Angeles City College). Harper, 1956. 591 pages, $5.00. Review: pp. 377–8, 1957.

A revision of a 1947 text.

Mathematics of Business. By Zant (Oklahoma State University). Prentice-Hall, 1956. 211 pages, $4.25.

Business Mathematics. By Mira and Hartmann (Manhattanville College of the Sacred Heart and St. John's University). Van Nostrand, 1955. 341 pages, $4.85.

Fundamentals of Business Mathematics, Second Edition. By Van Voorhis and Topp (Fenn College). Prentice-Hall, 1955. 452 pages, $6.95.

Revision of 1948 text.

Mathematics of Finance. By Mira and Hartmann (Manhattanville College of the Sacred Heart and St. John's University). Van Nostrand, 1954. 335 pages, $4.85.

REMEDIAL (INTERMEDIATE) ALGEBRA

Intermediate Algebra. By Bardell and Spitzbart (University of Wisconsin). Addison-Wesley, 1959. 274 pages, $4.75.

Introductory Mathematical Analysis. By Eaves and Wilson (University of Tennessee and Convair, Ft. Worth, Texas). Allyn and Bacon, 1958. 352 pages, $5.00. To be reviewed.

Intermediate Algebra for Colleges. By Fuller (Texas Technological). Van Nostrand, 1958. 258 pages, $3.90. To be reviewed.

Introducing Mathematics. By Helton (Central College). Wiley, 1958. 396 pages, $5.75.

Basic Mathematics. By Kaltenborn, Anderson and Kaltenborn (Memphis State College). Ronald Press, 1958. 350 pages, $4.75. To be reviewed.

The Structure of Arithmetic and Algebra. By Maria (Brooklyn College) Wiley, 1958. 294 pages, $5.90.

An elementary axiomatic development of the real number system written especially for teachers and nonscience students. To be reviewed.

Essential Mathematics for College Students. By Mueller (Towson, Md., State Teachers College). Prentice-Hall, 1958. 288 pages, $3.95.

A Modern Approach to Intermediate Algebra. By Patin (Wilson Junior College). G.P. Putnam's, 1958. 288 pages, $3.75. To be reviewed.

Preparatory Mathematics, Parts I and II. By Georges *et al.* (Wright Jr. College, *etc.*). Edwards, 1957. 208 and 216 pages, $4.00 and $4.50.

Fundamental Mathematics. By Miller (Ohio State University). Henry Holt, 1957. 288 pages, $3.50.

Intermediate Algebra, Second Edition. By Rees and Sparks (Louisiana State University and Texas Technological). McGraw-Hill, 1957. 306 pages, $3.90. Review: p. 381, 1957.

A revision of the 1951 text.

Intermediate Algebra for College Students. By White (Stevens Institute of Technology). Allyn and Bacon, 1957. 460 pages, $4.50.

Over 6000 exercises of standard type.

Basic Mathematics: A Workbook—Form B. By Keller and Zant (Purdue University and Oklahoma State University). Houghton Mifflin, 1956. 255 pages, $2.20.

A Review of Mathematics for College Students, Revised. By Lapp, Knight, Reitz and revised by Lapp (National Academy of Sciences-National Research Council). Scott, Foresman, 1956. 208 pages, $1.90 list.

A revision of the 1934 and 1942 workbook.

Elements of Algebra, Second Edition. By Levi (Columbia University). Chelsea, 1956. 168 pages, $3.25.

A path-breaking text.

Elements of Mathematics, Second Edition. By Roberts and Stockton (University of Connecticut) Addison-Wesley, 1956. 308 pages, $3.75. Review: pp. 605–6, 1957.

A tear-sheet workbook revision of a 1952 text.

Fundamental Mathematics. By Wade and Taylor (Florida State University). McGraw-Hill, 1956. 380 pages, $4.95.

First Course in Algebra for Colleges. By Adams (Santa Monica City College). Henry Holt, 1955. 217 pages, $3.50.

Basic Mathematics for General Education, Second Edition. By Trimble, Peck, and Bolser (Iowa State Teachers College, Ohio Wesleyan University). Prentice-Hall, 1955. 363 pages, $6.75.

Intermediate Algebra, Third Edition. By Adams (Santa Monica City College). Henry Holt, 1954. 366 pages, $4.50.

Elements of Mathematics. By Banks (George Peabody College for Teachers). Allyn and Bacon, 1954, 1956. 422 pages, $5.75.

Mathematics in Agriculture, Second Edition. By McGee (Texas A. and M. College). Prentice-Hall, 1954. 208 pages, $5.75.

A revision of a 1942 text.

Intermediate Algebra for College Students, Revised Edition. By Peterson (Portland State College). Harper, 1954. 369 pages, $3.75.

TRIGONOMETRY

Plane Trigonometry. By Niles (United States Naval Academy). Wiley, 1959. 284 pages, $3.95.

Plane Trigonometry, Second Edition. By Corliss and Berglund (University of Illinois. Navy Pier) Houghton Mifflin, 1958. 397 pages, $4.00.

Plane Trigonometry. By Rickey and Cole (Louisiana State University). Holt, née Dryden, 1958. 260 pages, $2.90. Review: pp. 639–40, 1958.

Elements of Plane Trigonometry. By Sharp (Emory University). Prentice-Hall, 1958. 298 pages, $4.95. To be reviewed.

Trigonometry, Plane and Spherical. By Hartley (University of Illinois). Odyssey, 1957. 374 pages, $3.80.

Plane Trigonometry, Third Edition. By Nelson and Folley (Wayne State University). Harper, 1956. 195 pages and 135 pages of tables, $4.00.

A revision of 1936 and 1943 texts.

Modern Trigonometry. By Rutledge and Pond (University of Tulsa and Technical Operations, Inc., Washington, D. C.). Prentice-Hall, 1956. 243 pages, $4.95. Review: pp. 55–6, 1957.

Practical Trigonometry. By Underwood and Woodward (Texas Technological). Houghton Mifflin, 1956. 251 pages, $3.25.

Modern Trigonometry. By Brixey and Andree (University of Oklahoma). Henry Holt, 1955. 224 pages, $4.00. Review: pp. 130–132, 1956.

Analytical trigonometry. A teaching guide is available.

Trigonometry. By Dubisch (Fresno State College). Ronald Press, 1955. 395 pages, $5.50, Review: pp. 458–9, 1955.

An outstanding development of analytical trigonometry of real numbers.

Trigonometry. By Perlin (Georgia Institute of Technology). International Textbook, 1955. 334 pages, $3.50.

Plane Trigonometry. By Spitzbart and Bardell (University of Wisconsin). Addison-Wesley, 1955. 205 pages, $3.75. Review: pp. 54–6, 1956.

Plane Trigonometry. By Wylie (University of Utah). McGraw-Hill, 1955. 381 pages, $4.50. Review: pp. 130–2, 1956.

Trigonometry. By Hart (University of Minnesota). D. C. Heath, 1954. 230 pages plus 130 pages of tables, $4.25.

Differs from the author's 1951 *College Trigonometry* only in chapters 1 to 4.

Trigonometry. By Vance (Oberlin College). Addison-Wesley, 1954. 158 pages, $3.75. Review: pp. 458–9, 1955.

Analytic Trigonometry.

COLLEGE ALGEBRA

College Algebra, Second Edition. By Peterson (Portland State College) Harper, 1958. 413 pages, $4.00.

A revision of a 1947 text.

College Algebra, Alternate Edition. By Richardson (Brooklyn College). Prentice-Hall, 1958. 544 pages, $5.95.

College Algebra. By Rietz, Crathorne, Peters (University of Illinois). Henry Holt, 1958. 387 pages, $4.50.

College Algebra, Fourth Edition. By Rosenbach, Whitman, Meserve and Whitman

(Carnegie Institute of Technology, Montclair State College and Johns Hopkins University). Ginn, 1958. 640 pages, $5.25. Review: p. 258, 1934 and p. 647, 1949.

A shorter edition (selections, not condensations) is available under the title *Essentials of College Algebra*, Second Edition. $4.50.

College Algebra. By Cameron and Browne (University of No. Carolina). Henry Holt, 1956. 390 pages, $4.75.

College Algebra for Freshmen. By Fuller (Texas Technological). Van Nostrand, 1956. 343 pages, $3.85.

College Algebra. By Kells (U. S. Naval Academy). Prentice-Hall, 1956. 366 pages, $5.75.

Applies axioms to natural numbers, then to integers, and then to rational numbers, real numbers and complex numbers.

College Algebra. By Apostle (Grinnell College). Henry Holt, 1955. 422 pages, $4.75. Review: pp. 192–3, 1955.

Algebra for College Students. By Whyburn and Daus (University of No. Carolina and U.C.L.A.). Prentice-Hall, 1955. 290 pages, $4.95.

Review: pp. 265–6, 1956.

College Algebra. By Rider (Washington University). Macmillan, 1955. 397 pages, $4.25.

Algebra for College Students. By Britton and Snively (University of Colorado). Rinehart, 1954. 537 pages, $4.50. Review: pp. 192–3, 1955.

College Algebra, Second Edition. By Keller (Purdue University) Houghton Mifflin, 1954. 471 pages, $4.50.

College Algebra, Third Edition. By Rees and Sparks (Louisiana State University and Texas Technological). McGraw-Hill, 1954. 455 pages, $4.75. Review: pp. 192–3, 1955.

A straightforward text; no "frills," but unusually fine graded problems.

COLLEGE ALGEBRA AND TRIGONOMETRY

Integrated Algebra and Trigonometry. By Fisher and Ziebur (Ohio State University). Prentice-Hall, 1958. 427 pages, $6.25. Review: p. 643, 1958.

College Algebra and Trigonometry, Second Edition. By Miller (Cooper Union School of Engineering). Wiley, 1955. 342 pages, $4.50.

College Algebra and Plane Trigonometry. By Spitzbart and Bardell (University of Wisconsin). Addison-Wesley, 1955. 408 pages, $5.25. Review: pp. 54–6, 1956.

Unified Algebra and Trigonometry. By Vance (Oberlin College). Addison-Wesley, 1955. 354 pages, $5.25. Review: pp. 54–6, 1956.

ANALYTIC GEOMETRY

Analytic Geometry. By Purcell (University of Arizona). Appleton-Century-Crofts, 1958. 289 pages, $4.50. To be reviewed.

Analytic Geometry of Three Dimensions, Seventh Edition. By Salmon (University of Dublin). Chelsea, 1958. 496 pages, $4.95. To be reviewed.

Brief Analytic Geometry, Third Edition. By Mason and Hazard (Purdue University). Ginn. 1957. 229 pages, $3.75. Review: p. 381, 1957.

Analytical Conics. By Spain (Sir John Cass College, London). Pergamon Press, 1957. 145 pages, $5.00. Review: p. 301, 1958.

A First Course in Analytic Geometry. By Ayre and Stephens (Western Illinois State College and Knox College). Van Nostrand, 1956. 224 pages, $4.10.

An interwoven presentation of plane and solid analytic geometry.

Analytic Geometry. By Rees (Louisiana State University). Prentice-Hall, 1956. 237 pages, $5.25.

New Analytic Geometry, Revised Edition. By Smith, Gale, Neelley (Carnegie Institute of Technology). Ginn, 1956. 346 pages, $4.00.

Analytic Geometry, Second Edition. By Underwood and Sparks (Texas Technological). Houghton Mifflin, 1956. 282 pages, $3.75.

Analytic Geometry. By Love and Rainville (University of Michigan). Macmillan, 1955. 302 pages, $4.25.

Analytic Geometry. By McCoy and Johnson (Smith College). Rinehart, 1955. 301 pages $3.50. Review: pp. 675–6, 1956.

Analytic Geometry, Second Edition. By Middlemiss (Washington University). McGraw-Hill, 1955. 310 pages, $4.25. Review: pp. 675–6, 1956.

Analytic Geometry. By Sisam and Atchison (Colorado College and Rich Electronic Computer Center). Henry Holt, 1955. 292 pages, $4.00.

Analytic Geometry, Third Edition. By Steen and Ballou (Allegheny College and Middlebury College). Ginn, 1955. 251 pages, $4.00.

Analytic Geometry. By Fuller (Texas Technological). Addison-Wesley, 1954. 205 pages, $4.25. Review: pp. 196–7, 1955.

Conic Sections, Sixth Edition. By Salmon (University of Dublin). Chelsea, 1954. 415 pages, $3.25 cloth; $1.94 paper-bound.

A classic.

Analytic Geometry, Second Edition. By Smith, Salkover and Justice (University of Cincinnati). Wiley, 1954. 306 pages, $4.00. Review: pp. 196–7, 1955.

Covers both the applied and the theoretical sides of analytical geometry.

INTEGRATED FRESHMAN MATHEMATICS

A Modern Introduction to College Mathematics. By Rose (University of Massachusetts). Wiley, 1959. 548 pages, $6.50. To be reviewed.

Introductory College Mathematics. By Wade (Florida State University). Wiley, 1959. 344 pages, $5.50.

Introduction to Mathematical Analysis with Applications to Problems of Economics. By Daus and Whyburn (U.C.L.A. and University of No. Carolina). Addison-Wesley, 1958. 244 pages, $6.50.

Fundamentals of Mathematics. By Richardson (Brooklyn College). Macmillan, 1958. 507 pages, $6.50. To be reviewed.

Mathematics for Science and Engineering. By Alger (General Electric Company). McGraw-Hill, 1957. 360 pages, $6.95, Text edn. $5.50.

Very broad coverage, derived partially from the old Steinmetz books. Review: p. 57, 1958.

Introduction to Finite Mathematics. By Kemeny, Snell, and Thompson. (Dartmouth and Ohio Wesleyan University). Prentice-Hall, 1957. 372 pages, $5.95. Review: pp. 688–9, 1957.

A true trail-blazer, designed particularly as a basic text for behavioral science students, but is being used elsewhere as well. Includes no calculus.

Introductory College Mathematics. By Wagner (University of Massachusetts). McGraw-Hill, 1957. 430 pages, $5.25. Review: pp. 461–2, 1958.

Mathematical Analysis. By Camp (Macalester College). D. C. Heath, 1956. 610 pages, $6.75.

A Modern Introduction to Mathematics. By Freund (Arizona State). Prentice-Hall, 1956. 543 pages, $6.95.

Principles of Mathematics. By Allendoerfer and Oakley (University of Washington and Haverford). McGraw-Hill, 1955. 448 pages, $5.75. Review: pp. 435–9, 1956.

Another trail-blazer.

Basic Mathematics for Science and Engineering. By Andres, Miser and Reingold (deceased, U.S.A.F., Illinois Institute of Technology). Wiley, 1955. 846 pages, $7.50.

Introductory Mathematical Analysis. By Georges (Wright Junior College). J. W. Edwards, 1955. 624 pages, $6.00.

Introduction to College Mathematics. By Hill and Linker (University of No. Carolina). Henry Holt, 1955. 428 pages, $6.50.

Mathematics and Measurements. By Rassweiler, Merrill, and Harris (University of Minnesota). Row, Peterson, 1955. 251 pages, $4.50.

Basic Mathematics for General Education, Second Edition. By Trimble, Peck, and Bolser (Iowa State Teachers College, Ohio Wesleyan, Govt. Consultant). Prentice-Hall, 1955. 363 pages, $6.75.

A First Year of College Mathematics, Second Edition. By Brink (University of Minnesota). Appleton-Century-Crofts, 1954. 725 pages, $5.75.

Fundamentals of College Mathematics. By Brixey and Andree (University of Oklahoma). Holt, 1954. 609 pages, $6.95. Review: pp. 193–5, 1955.

Introductory College Mathematics. By Jaeger and Bacon (Pomona College and Stanford University). Harper, 1954. 382 pages, $4.75.

Introductory College Mathematics. By Leonhardy (Stephens College). Wiley, 1954. 459 pages, $5.50.

Introduction to College Mathematics, Second Edition. By Newsom and Eves (New York University and University of Maine). Prentice-Hall, 1954. 408 pages, $6.95. Review: 724–5, 1954.

CALCULUS WITH ANALYTIC GEOMETRY

Analytic Geometry and Calculus. By Hart (University of Minnesota). D. C. Heath, 1957. 716 pages, $7.00.

Calculus with Analytic Geometry. By Johnson and Kiokemeister (Smith College and Mt. Holyoke College). Allyn and Bacon, 1957. 650 pages, $8.25. Review: pp. 640–1, 1958.

Analytic Geometry and Calculus. By Peterson (Portland State College). Harper, 1955. 456 pages, $6.00. Review: pp. 674–5, 1956.

Introductory Calculus with Analytic Geometry. By Begle (Yale University). Holt, 1954. 304 pages, $5.25. Review: pp. 54–6, 1955.

CALCULUS

Calculus. By Leighton (Carnegie Institute of Technology). Allyn and Bacon, 1958. 416 pages, $6.95. To be reviewed.

An Analytical Calculus, Volume IV. By Maxwell (Cambridge University). Cambridge University Press, New York, 1958. 272 pages, $4.00. Review: pp. 536–7, 1958.

Calculus, Second Edition. By Smith, Salkover and Justice (University of Cincinnati). Wiley, 1958. 520 pages, $6.50. Review: p. 377, 1958.

Elements of the Differential and Integral Calculus, New Revised Edition. By Granville, Smith and Longley (Yale University). Ginn, 1957. 556 pages, $5.75.

Calculus. By Britton (University of Colorado). Rinehart, 1956. 584 pages, $6.50. Review: pp. 462–3, 1958.

Calculus. By Morrill (Johns Hopkins University). Van Nostrand, 1956. 537 pages, $6.00.

Differential and Integral Calculus, Second Edition. By Bacon (Stanford University). McGraw-Hill, 1955. 547 pages, $6.50. Review: pp. 202–3, 1956.

Calculus. By Hart (University of Minnesota). D. C. Heath, 1955. 626 pages, $6.75. Review: p. 202–3, 1956.

First Course in Calculus. By Cooley (New York University). Wiley, 1954. 643 pages, $6.50. Review: pp. 52–4, 1955.

Differential and Integral Calculus. By Love and Rainville (University of Michigan). Macmillan, 1954. 526 pages, $5.95. Review: pp. 454–8, 1955.

An Analytical Calculus. By Maxwell (Cambridge University). Cambridge University Press, New York, 1954. Vol. I—200 pp.; Vol. II—200 pp.; Vol. III, 200 pp., $3.50 each.

Calculus: A Modern Approach. By Menger (Illinois Institute of Technology). Ginn, 1954. 372 pages, $5.50. Review: pp. 483–92, 1954.

Calculus. By Merriman (University of Cincinnati). Holt, 1954. 626 pages, $6.95. Review: pp. 454–8, 1955.

Calculus, Third Edition. By Sherwood and Taylor (U.C.L.A.). Prentice-Hall, 1954. 579 pages, $7.95. Review: pp. 454–8, 1955.

POST-CALCULUS ENGINEERING MATHEMATICS

Engineering Mathematics. By Gaskell (Boeing Scientific Research Labs). Holt, née Dryden, 1958. 512 pages, $7.75. To be reviewed.

Applied Mathematics for Engineers and Physicists, Second Edition. By Pipes (U.C.L.A.). McGraw-Hill, 1958. 723 pages, $8.75. Revision of 1946 book. To be reviewed.

Mathematics of Physics and Modern Engineering. By Sokolnikoff and Redheffer (U.C.L.A.). McGraw-Hill, 1958. 828 pages, $9.50. To be reviewed.

Modern Mathematics for the Engineer. By Beckenbach (U.C.L.A.). McGraw-Hill, 1956. 536 pages, $7.50. To be reviewed.

Engineering Analysis (A Survey of Numerical Procedures). By Crandall (M.I.T.). McGraw-Hill, 1956. 417 pages, $9.50. To be reviewed.

Engineering Mathematics. By Miller (New York University). Rinehart, 1956. 417 pages, $6.50. Review: pp. 281–2, 1957.

Mathematics of Engineering Systems (*Linear and Nonlinear*). By Lawden (College of Technology, Birmingham, England). Wiley, 1955. 380 pages, $5.75.

Higher Mathematics for Students of Chemistry and Physics. By Mellor. Dover, 1955. 641 pages, $2.25.

Advanced Mathematics for Engineers, Third Edition. By Reddick and Miller (Syracuse University and Cooper Union School of Technology). Wiley, 1955. 548 pages, $6.50.

Engineering Analysis. By Ver Planck and Teare (Carnegie Institute of Technology). Wiley, 1954. 344 pages, $6.00.

ADVANCED CALCULUS

Mathematical Analysis, A Modern Approach to Advanced Calculus. By Apostol (California Institute of Technology). Addison-Wesley, 1957. 553 pages, $9.00. Review: pp. 463–4, 1958.

Designed primarily for mathematics majors.

Elements of Pure and Applied Mathematics (*International Series*). By Lass (California Institute of Technology). McGraw-Hill, 1957. 491 pages, $7.50. Review: p. 517, 1957.

Advanced Real Calculus. By Miller (New York University). Harper, 1957. 185 pages, $5.00. Review: p. 519, 1957.

Advanced Calculus (*International Series in Pure and Applied Mathematics*). By Buck (University of Wisconsin). McGraw-Hill, 1956. 432 pages, $8.50. Review: pp. 754–5, 1956.

Advanced Calculus—An Introduction to Classical Analysis. By Brand (University of Houston), Wiley. 1955. 574 pages, $8.50. Review: p. 786, 1958.

Advanced Calculus. By Taylor (University of California). Ginn, 1955. 799 pages, $8.75. Review: pp. 733–4, 1956.

The Integral Calculus (two volumes). By Edwards (Queen's College). Chelsea, 1954. 1,922 pages, $15.00.

A comprehensive, detailed text, with many examples.

Advanced Calculus, New Edition. By Woods. Ginn, 1954. 406 pages, $6.25.

DIFFERENTIAL EQUATIONS

Lectures on Ordinary Differential Equations. By Hurewicz (Formerly M.I.T.) Wiley, 1958. 122 pages, $5.00. To be reviewed.

Ordinary Differential Equations. By Kaplan (University of Michigan). Addison-Wesley, 1958. 534 pages, $9.50.

Emphasis on input-output analysis and applications to systems engineering problems.

Elementary Differential Equations. By Rainville (University of Michigan). Macmillan, 1958. 449 pages, $5.50. To be reviewed.

A Short Course in Differential Equations. By Rainville (University of Michigan). Macmillan, 1958. 259 pages, $4.50. To be reviewed.

Applied Differential Equations. By Spiegel (Rensselaer Polytechnic Institute). Prentice-Hall, 1958. 381 pages, $6.95. To be reviewed.

Differential Equations: Geometric Theory. By Lefschetz (Princeton University). Interscience, 1957. 374 pages, $10.50. Review: pp. 784–5, 1958.

Differential Equations Applied in Science and Engineering. By Wayland (California Institute of Technology). Van Nostrand, 1957. 353 pages, $7.50. To be reviewed.

Theory of Ordinary Differential Equations. By Burkill (University of Cambridge). Interscience, 1956. 106 pages, $1.75.

Elementary Differential Equations. By Martin and Reissner. Addison-Wesley, 1956. 260 pages, $6.50.

Differential Equations, Third Edition. By Reddick and Kibbey (Syracuse University). Wiley, 1956. 304 pages, $4.50.

Theory of Ordinary Differential Equations (*International Series*). By Coddington and Levinson. (U.C.L.A. and M.I.T.). McGraw-Hill, 1955. 429 pages, $8.50.

Differential Equations, Second Edition. By Ford (Formerly of Illinois Institute of Technology). McGraw-Hill, 1955. 288 pages, $5.25.

Introduction to Differential Equations of Physics. By Hopf. Dover, 1955. 160 pages, $1.25.

Differential Equations. By Steen (Allegheny College). Ginn, 1955. 337 pages, $4.75.

Differential Equations with Applications. By Betz, Burcham and Ewing (University of Missouri). Harper, 1954. 310 pages, $4.50. Review: pp. 130–1, 1955.

Ordinary Differential Equations. By Ince. Dover, 1954. 558 pages, $2.55.

A First Course in Ordinary Differential Equations. By Langer (University of Wisconsin). Wiley, 1954. 249 pages, $4.50. Review: p. 437, 1954.

Existence Theorems for Ordinary Differential Equations. By Murray and Miller (Columbia University and New York University). Interscience, 1954. 192 pages, $6.00. Review: p. 668, 1955.

PARTIAL DIFFERENTIAL EQUATIONS

Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations. By Noble (Royal College of Science and Technology, Glasgow). Pergamon Press, 1958. 280 pages, $10.00.

Elements of Partial Differential Equations. By Sneddon (University of Glasgow). McGraw-Hill, 1957. 345 pages, $7.50. Review: pp. 216–7, 1958.

The Hypercircle in Mathematical Physics. By Synge (Dublin Institute). Cambridge University Press, 1957. 440 pages, $13.50. Review: pp. 217–8, 1958.

Partial Differential Equations of Mathematical Physics. By Bateman. Dover, 1955. 522 pages, $4.95. Not currently available.

Lectures on Partial Differential Equations. By Petrovsky (Akademiia Nauk, Moscow). Interscience, 1955. 255 pages, $6.50.

Partial Differential Equations of Mathematical Physics. By Webster. Dover, 1955. 440 pages, $1.98.

Partial Differentiation. By Gillespie (University of Glasgow). Interscience, 1954. 113 pages, $1.75.

SERIES

Introduction to Fourier Analysis and Generalized Functions. By Lighthill (University of Manchester). Cambridge University Press, 1958. 88 pages, $3.50. To be reviewed.

Orthogonal Functions. By Sansone (University of Florence). Interscience, 1958. 420 pages, Approx. $13.50.

Expansions in series of orthogonal functions, together with preliminary notions of Hilbert space, followed by expansions in Fourier series, in series of Legendre polynomials, and spherical harmonics, and expansions in Laguerre and Hermite series.

Infinite Series and Sequences. By Knopp. Dover, 1956. 186 pages, $1.75.

Trigonometrical Series. By Zygmund. Dover, 1955. 329 pages, $1.50.

Contains analyses of trigonometric, orthogonal, Fourier systems of functions, with descriptions of summability of Fourier series, proximation theory, conjugate theory, divergence of Fourier series.

Introduction to the Theory of Fourier's Series and Integrals. By Carslaw. Dover, 1955. 368 pages, $2.00.

A revision of the author's 1930 book giving an historical introduction, rational and irrational numbers, infinite series, functions of a single variable, definite integral, Fourier series, Fourier integrals, harmonic analysis, and periodogram analysis.

Infinite Series. By Hyslop (University of Glasgow). Interscience, 1954. 133 pages, $1.55.

Mainly real series, but some work on complex series and on infinite products.

APPLIED MATHEMATICS

Operational Mathematics, Second Edition. By Churchill (University of Michigan). McGraw-Hill, 1958. 440 pages, $7.00. To be reviewed.

Methods of Mathematical Physics. Volume II. By Courant and Hilbert (Institute of Mathematical Science). Interscience, in preparation.

Linear Operators in Two Parts—Part I, General Theory, Part II, Spectral Theory. By Dunford and Schwartz. Assisted by Bade and Bartle (Yale University). Interscience, 1958. 872 pages, Part I, $23.00. Part II, in preparation. To be reviewed.

Celestial Mechanics. By Finlay-Freundlich (St. Andrews, Fife). Pergamon Press, 1958. $7.50.

Surveys in Applied Mathematics. Volume I (*Midwest Research Institute Series*) deals with Elasticity and Plasticity. By Goodier and Hodge (Stanford University and Illinois Institute of Technology). Wiley, 1958. 152 pages, $6.25. To be reviewed.

Surveys in Applied Mathematics. Volume II (*Midwest Research Institute Series*) deals with Dynamics and Nonlinear Mechanics. By Leimanis and Minorsky (University of British Columbia and Aix-en-Provence, France). Wiley, 1958. 206 pages, $7.75. To be reviewed.

Surveys in Applied Mathematics. Volume III (*Midwest Research Institute Series*) discusses Mathematical Aspects of Subsonic and Transonic Gas Dynamics. By Bers (New York University). Wiley, 1958. 164 pages, $7.75. To be reviewed.

Surveys in Applied Mathematics. Volume IV (*Midwest Research Institute Series*) deals with some aspects of analysis and probability. By Kaplansky, Hewitt and Fortet (University of Chicago and University of Washington). Wiley, 1958. 243 pages, $9.00. To be reviewed.

Surveys in Applied Mathematics. Volume V (*Midwest Research Institute Series*). This volume deals with Numerical Analysis and Partial Differential Equations. By Forsythe and Rosenbloom (Stanford University and University of Minnesota). Two separate papers in one binding. Wiley, 1958. 204 pages, $7.50. To be reviewed.

Calculus of Variations and Its Applications: Proceedings of Symposia in Applied Mathematics Volume VIII. By Graves, Editor. McGraw-Hill for American Mathematical Society, 1958. 215 pages, $7.50. Review: p. 648, 1958.

Reference volume for graduate students and researchers; papers by leading scholars, presented at the symposium held in Chicago in 1956.

Modern Geometrical Optics. By Herzberger (Eastman Kodak Co.). Interscience, 1958. 516 pages, $15.00.

An Introduction to Advanced Dynamics. By McCuskey (Case Institute of Technology). Addison-Wesley, 1959. 269 pages, $8.50.

Mathematics and Wave Mechanics. By Atkin (Northern Polytechnic, London, England). Wiley, 1957. 348 pages, $6.00. To be reviewed.

Integral Equations and Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology. By Mikhlin, translated from Russian by A. H. Armstrong. Pergamon Press, 1957. 338 pages, $12.50. Review; pp. 644–5, 1958.

Development and Meaning of Eddington's Fundamental Theory. By Noel B. Slater (University of Leeds). Cambridge University Press, New York, 1957. 308 pages, $7.50. Review: p. 134, 1958.

Water Waves: The Mathematical Theory with Applications. By Stoker (New York University). Interscience, 1957. 595 pages, $12.75.

Integral Equations. By Tricomi (University of Turin, Italy). Interscience, 1957. 246 pages, $7.75.

Surveys in Mechanics. By Batchelor (Cambridge University). Cambridge University Press, New York, 1956. 432 pages, $9.50.

Vibration Analysis Tables. By Bishop and Johnson (Cambridge University). Cambridge University Press, New York, 1956. 64 pages, $2.00.

Integral Functions. By Cartwright (Cambridge University). Cambridge University Press, New York, 1956. 130 pages, $3.50.

Electricity. By Coulson (Oxford University). Interscience, 1956. 268 pages, $1.95.

Principles and Techniques of Applied Mathematics. By Friedman (University of California, Berkeley). Wiley, 1956. 315 pages, $8.00.

Principally devoted to linear operators.

The Mathematics of Physics and Chemistry, Second Edition. By Margenau and Murphy (Yale University and New York University). Van Nostrand, 1956. 604 pages, $7.25.

Special Functions of Physics and Chemistry. By Sneddon (University College of North Staffordshire). Interscience, 1956. 171 pages, $1.95.

Discusses the hypergeometric functions, the functions of Legendre, Bessel, Hermite and Laguerre, and the Dirac delta Function.

Mathematical Theory of Elasticity, Second Edition. By Sokolnikoff. (U.C.L.A.). McGraw-Hill, 1956. 476 pages, $9.75.

The Structure of Turbulent Shear Flow. By Townsend (Cambridge University). Cambridge University Press, New York, 1956. 320 pages. $7.50.

Waves: A Mathematical Account of the Common Types of Wave Motion. By Coulson (Oxford University). Interscience, 1955. 171 pages, $1.75.

Integration. By Gillespie (University of Glasgow). Interscience, 1955. 140 pages, $1.55.

First four chapters are devoted to an elementary account of integration. The book then takes up the Riemann integral, infinite integrals (particularly gamma and beta functions), and the Riemann double integral.

Plane Waves and Spherical Means. Applied to Partial Differential Equations. By John (New York University). Interscience, 1955. 180 pages, $5.50.

Theory of functions of a complex variable and advanced calculus are the only prerequisites for this volume.

Hydrodynamic Stability. By Lin. Cambridge University Press, New York, 1955. 160 pages, $4.75.

Physical Mathematics. By Page (National Bureau of Standards). Van Nostrand, 1955. 329 pages, $6.00.

Introduction to Relaxation Methods. By Shaw. Dover, 1955. 400 pages, $2.45.

Static and Dynamic Electron Optics. By Sturrock (Cambridge University). Cambridge University Press, New York, 1955. 240 pages, $6.50.

Linearized Theory of Steady High-Speed Flow. By Ward (College of Aeronautics, Cranfield). Cambridge University Press, New York, 1955. 248 pages, $6.00.

Introduction to Elliptic Functions with Applications. By Bowman (College of Technology, Manchester, England). Wiley, 1954. 115 pages, $2.75.

Discusses Jacobian elliptic functions with the emphasis directed to conformal mapping of polygons.

Thermodynamics. By Fermi. Dover, 1954. 160 pages, $1.75.

Thermodynamic systems, first and second laws of thermodynamics, entropy, thermodynamic potential, phase rule, reversible electric cell, *etc.*

Foundations of Potential Theory. By Kellogg. Dover, 1954. 384 pages, $1.98.

Linear Integral Equations. By Lovitt. Dover, 1954. 253 pages, $1.60.

Linear Equations in Applied Mechanics. By Purday (Harland and Wolff, Ltd.). Interscience, 1954. 254 pages, $4.00.

COMPLEX VARIABLES

Theory of Functions (two volumes). By Caratheodory (University of Munich). Chelsea, 1958 and 1954. 310 and 220 pages, $4.95 each or $9.90 the set.

Functions of Complex Variables. By Franklin (M.I.T.) Prentice-Hall, 1958. 246 pages, $7.25. To be reviewed.

Multivalent Functions. By Hayman (Imperial College of Science and Technology, London). Cambridge University Press, New York, 1958. 168 pages, $4.00. To be reviewed.

Elements of the Theory of Functions. By Knopp. Dover, 1955. 160 pages, $1.35.

Transform Calculus with an Introduction to Complex Variables. By Scott (University of Illinois). Harper, 1955. 330 pages, $7.50. Review: pp. 351–3, 1956.

REAL VARIABLES

Functions of Real Variables and Functions of a Complex Variable. By Osgood (Harvard University). Two volumes in one. Chelsea, 1958. 676 pages, $4.95. To be reviewed.

Foundations of Analysis. By Landau (Goettingen University). Chelsea, 1957. 148 pages, $3.50.

Reprint of 1951 edition.

The Theory of Functions of Real Variables, Second Edition (*International Series*). By Graves (University of Chicago). McGraw-Hill, 1956. 375 pages, $7.50.

Intermediate Analysis. By Olmsted (University of Minnesota). Appleton-Century-Crofts, 1956. 305 pages, $6.00. To be reviewed.

Contributions to the Founding of the Theory of Transfinite Numbers. By Cantor. Dover, 1955. 211 pages, $1.25.

VECTOR ANALYSIS

Vector Analysis. By Brand (University of Houston). Wiley, 1957. 282 pages, $6.00. Review: pp. 297–8, 1958.

Vector and Tensor Analysis. By Coburn (University of Michigan). Macmillan, 1955. 341 pages, $7.00.

Vector Analysis. By Newell (Office of Naval Research). McGraw-Hill, 1955. 216 pages, $5.50. Review: p. 505, 1956.

Vector and Tensor Analysis. By Hay (University of Michigan). Dover, 1954. 208 pages, $1.75.

Vector Methods. By Rutherford (University of St Andrews). Interscience, 1954. 143 pages, $1.75.

MODERN ABSTRACT ALGEBRA

The Theory of Groups. By Hall (Ohio State University). Macmillan, 1959. 416 pages, $8.75.

Selections from Modern Abstract Algebra. By Andree (University of Oklahoma). Holt, 1958. 212 pages, $6.50. Review: p. 158, 1959.

Matrix Calculus, Second Edition Revised. By Bodewig. Interscience, in preparation.

Introduction to Difference Equations. By Goldberg (Oberlin College). Wiley, 1958. 260 pages, $6.75. To be reviewed.

Finite-Dimensional Vector Spaces. By Halmos (University of Chicago). Van Nostrand, 1958. 208 pages, $5.00. To be reviewed.

Elementary Matrix Algebra. By Hohn (University of Illinois). Macmillan, 1958. 352 pages, $7.50. To be reviewed.

Elements of Modern Abstract Algebra. By Miller (New York University). Harper, 1958. 188 pages, $5.00. To be reviewed.

Introduction to Functional Analysis. By Taylor (University of California). Wiley, 1958. 423 pages, $12.50. To be reviewed.

Commutative Algebra, Volume I. By Zariski and Samuel (Harvard University and University of Clermont-Ferrand). Van Nostrand, 1958. 329 pages, $6.95. To be reviewed.

The Theory of Groups, Second Edition. By Zassenhaus (McGill University). Chelsea, 1958. 271 pages, $6.00.

Introduction to the Theory of Equations. By Conkwright (University of Iowa). Ginn, 1957. 222 pages, $4.50.

Linear Algebra for Undergraduates. By Murdoch (University of British Columbia). Wiley, 1957. 239 pages, $5.50. Review: pp. 300–301, 1958.

Vector Spaces and Matrices. By Thrall and Tornheim (University of Michigan and California Research Corporation). Wiley, 1957. 318 pages, $6.75. To be reviewed.

Determinants and Matrices. By Aitken (University of Edinburgh). Interscience, 1956. 151 pages, $1.65.

Fundamental Concepts of Higher Algebra. By Albert (University of Chicago). University of Chicago Press, 1956. 216 pages, $6.50. Review: p. 602, 1957.

Theory of Groups of Finite Order. By Burnside. Dover, 1955. 447 pages, $2.45.
A new printing of Burnside's 1911 text.

Introduction to the Theory of Groups of Finite Order. By Carmichael (University of Illinois).

Dover, 1955. 447 pages, $2.00.

A reprint of the earlier book.

The Theory of Groups (two volumes). By Kurosh (University of Moscow). Chelsea, 1955 and 1956. 271 and 308 pages, $4.95 each or $9.90 the set.

Introduction to Modern Algebra and Matrix Theory. By Schreier and Sperner (Hamburg University). Chelsea, 1955. 386 pages, $6.00.

Introduction to Modern Algebra and Matrix Theory. By Beaumont and Ball (University of Washington and Alabama Polytechnic Institute). Rinehart, 1954. 331 pages, $6.00. Review: pp. 499–500, 1955.

Theory of Equations. By MacDuffee (University of Wisconsin). Wiley, 1954. 120 pages, $3.75. Review: pp. 723–4, 1954.

Certain basic concepts from abstract algebra are woven into the usual work in *Theory of Equations.*

THEORY OF NUMBERS

Introduction to the Theory of Numbers. By Dickson. Dover, 1958. 183 pages, $1.65. Review: p. 650, 1958.

A reprint of the original Dickson volume.

Elementary Number Theory. By Landau (Goettingen University). Chelsea, 1958. 256 pages, $4.95. To be reviewed.

With exercises by Professors Bateman and Kohlbecker.

Introduction to Diophantine Approximation. By Cassels (Cambridge University). Cambridge University Press, New York, 1956. 170 pages, $4.00. Review: pp. 465–6, 1958.

Topics in Number Theory, Volumes I and II. By LeVeque (University of Michigan). Addison-Wesley, 1956. Vol. I, 198 pages, $6.50. Vol. II, 270 pages, $7.50. Review: pp. 445–7, 1957.

Volume I is excellent for the usual first course in Number Theory. Volume II brings together advanced topics in analytic number theory, many of which are not available elsewhere in English texts.

Irrational Numbers. By Niven (University of Oregon). Wiley, 1956. 164 pages, $3.00. Review: pp. 606–7, 1957.

One of the excellent Carus Monograph Series.

The Number-System. By Thurston (University of Bristol, England). Interscience, 1956. 144 pages, $3.00. Review: pp. 295–6, 1958.

The number system is derived formally from the Peano axioms, using Cauchy sequences.

The Theory of Numbers. By Jones (University of Colorado). Rinehart, 1955. 143 pages, $3.75. Review: pp. 52–3, 1956.

An unusually interesting book in elementary number theory with an emphasis on the "discovery method."

Elements of Number Theory. By Vinogradov. Dover, 1955. 227 pages, $1.60.

An Introduction to the Theory of Numbers. By Vinogradov (Akademiia Nauk, Moscow) translated by Popova (University of Aberdeen). Pergamon Press, 1955. 155 pages, $3.00.

Elementary Theory of Numbers (*International Series*). By Griffin (Brooklyn College). McGraw-Hill, 1954. 203 pages, $5.00. Review: p. 132, 1955.

The Method of Trigonometrical Sums in the Theory of Numbers. By Vinogradov (Akademiia Nauk, Moscow). Interscience, 1954. 190 pages, $5.50.

A translation of the later of Vinogradov's two monographs on the subject.

NUMERICAL ANALYSIS

The Calculus of Finite Differences, Fourth Edition. By Boole (Queen's College). Chelsea, 1958. 348 pages, $4.95. To be reviewed.

Linear Programming. By Gass. McGraw-Hill, 1958. 233 pages, $6.50. To be reviewed.

Numerical Analysis. By Kunz (Schlumberger Corp). McGraw-Hill, 1957. 575 pages, $8.00. Review: pp. 464–5, 1958.

Difference Methods for Initial-Value Problems. By Richtmyer (New York University). Interscience, 1958. 250 pages, $7.25. To be reviewed.

Designed for use with today's computers.

Introduction to Numerical Analysis (*International Series*). By Hildebrand (M.I.T.). McGraw-Hill, 1956. 511 pages, $8.50. Review: pp. 128–30, 1957.

Applied Analysis. By Lanczos (Dublin Institute for Advanced Studies). Prentice-Hall, 1956. 608 pages, $6.75. Review: pp. 447–8, 1957.

Methods in Numerical Analysis. By Nielsen (Naval Ordnance Plant, Indianapolis, Indiana). Macmillan, 1956. 382 pages, $6.90. Review: p. 594, 1956.

Numerical Analysis. By Kopal (University of Manchester, England). Wiley, 1955. 556 pages, $12.00. Review: pp. 127–8, 1957.

Numerical Solutions of Differential Equations. By Levy and Baggot (University of Illinois). Dover, 1955. 238 pages, $1.75.

Practical Analysis, Graphical and Numerical Methods. By Willers. Dover, 1955. 422 pages, $2.00.

An Introduction to the Calculus of Finite Differences. By Richardson (Bucknell University). Van Nostrand, 1954. 142 pages, $4.00. Review: pp. 132–3, 1955.

COMPUTERS

Programming Business Computers. By McCracken, Weiss and Lee (General Electric Company). Wiley, 1959. 504 pages, $8.00.

A general introduction to business computers.

Programming the IBM 650 Computer. By Andree (University of Oklahoma). Holt, 1958. 120 pages, $2.95. To be reviewed.

Discusses high-speed programming techniques (flow charting) in general, using the IBM 650 commands. A classroom text with problems.

Mathematics and Logic for Digital Devices. By Culbertson (California State Polytechnic College). Van Nostrand, 1958. 230 pages, $4.85. To be reviewed.

Handbook of Automation, Computation and Control, Volume I. By Grabbe, Ramo and

Wooldridge (Ramo-Wooldridge Corporation). Wiley, 1958. 1020 pages, $17.00. To be reviewed.

Provides specific coverage on aspects of mathematics as applied to control; a compilation of the mathematics of digital computers; the latest techniques and comparisons of different techniques involving computers; an all-in-one treatment of servo theory and operations research; all necessary material on information theory and transmission.

Elementary Mathematical Programming. By Metzger (General Motors Institute). Wiley, 1958. 246 pages, $5.95.

An introduction to mathematical programming with special emphasis on the interests of business and industrial people.

Your Fingers: Nature's Digital Computer. By Blake. William-Frederick Press, 1957. 24 pages, $2.00.

An Introduction to Automatic Computers: A Systems Approach for Business, Second Edition. By Chapin (Stanford Research Institute). Van Nostrand, 1957. 525 pages, $6.75.

Digital Computer Programming. By McCracken (General Electric Company). Wiley, 1957. 253 pages, $6.50. Review: pp. 132–3, 1958.

A general introduction to computers which contains the practical details necessary to work with specific machines. Discusses high speed computer programming in general using the operation codes of a mythical computer TYDAC.

Computing With Desk Calculators. By Varner (Flight Simulation Lab., Convair, San Diego). Rinehart, 1957. 108 pages, $2.00. Review: p. 692, 1957.

The Preparation of Programs for an Electronic Digital Computer, Second Edition. By Wilkes, Wheeler and Gill (Cambridge University, England). Addison-Wesley, 1957. 256 pages, $7.50. Review: pp. 719–20, 1958.

Arithmetric Operations in Digital Computers. By Richards (IBM Engineering Laboratory). Van Nostrand, 1955. 397 pages, $8.00.

TABLES

Six Figure Logarithms, Antilogarithms and Trigonometrical Functions. By Attwood (Trade School, Ford Motor Co. Ltd., Dagenham). Pergamon Press, 1958. 132 pages, $1.25.

Burington Tables. By Burington (Bureau of Ordnance, Navy Dept.). McGraw-Hill née Handbook Publishers.

Mathematical Tables and Formulae. By Camm (Cambridge University). Philosophical Library, 1958. 144 pages, $2.75. To be reviewed.

Handbook of Calculus, Difference and Differential Equations. By Cogan and Norman (Sarah Lawrence, Dartmouth). Prentice-Hall, 1958. 263 pages, $4.50. To be reviewed.

CRC Standard Mathematical Tables, Eleventh Edition revised. Chemical Rubber. 1959. 480 pages, $3.00: To be reviewed.

Tables of Integrals and Other Mathematical Data. By Dwight (M.I.T.). Macmillan, 1957. 288 pages, $3.00.

A Short Table of Integrals, Fourth Edition. By Peirce and Foster. (Polytechnic Institute of Brooklyn). Ginn, 1956. 196 pages, $2.85.

Tables of Functions with Formulae and Curves. By Jahnke and Emde. Dover, 1955. 382 pages, $2.00.

Bateman Tables Project. Erdelyi (Cal. Tech), Editor. The "Bateman Tables Project," (sponsored by California Institute Technology) provides in five convenient volumes the essence of the enormous compilation of functions and *transforms* materials gathered over a period of many years by the late Harry Bateman.

Tables of Integral Transforms, Volume I, McGraw-Hill, 1954. 391 pages, $7.50.

Tables of Integral Transforms, Volume II, McGraw-Hill, 1954. 451 pages, $8.00.

Higher Transcendental Functions, Volume III, McGraw-Hill, 1955. 312 pages, $6.50.

Four Place Tables of Transcendental Functions. By Flügge (Stanford University). Pergamon Press, 1954. 136 pages, $5.00.

Formulas and Theorems for the Functions of Mathematical Physics. By Magnus and Oberhettinger (New York University and University of Wisconsin). Chelsea, 1954. 182 pages, $3.90.

Reprint of 1949 edition.

The Biometrika Tables for Statisticians, Volume I. By Pearson and Hartley (University College, London). Cambridge University Press, New York, 1954. 250 pages, $5.00.

GEOMETRY

Plane Geometry for Colleges. By Adams (Santa Monica City College). Holt, 1958. 214 pages, $3.50. Review: p. 787, 1958.

Modern Geometry: An Integrated First Course. By Adler (New York University). McGraw-Hill, 1958. 215 pages, $6.00. To be reviewed.

Basic Geometry. By Birkhoff and Beatley (Harvard University). Chelsea, 1958. 294 pages, $3.95. To be reviewed.

A corrected edition of the 1940 classic.

College Plane Geometry. By Hemmerling (Bakersfield College) Wiley, 1958. 310 pages, $4.95. To be reviewed.

An Introduction to Euclidean Geometry. By Eaves and Robinson (University of Kentucky and Alabama Polytechnic Institute). Addison-Wesley, 1957. 327 pages, $4.25. To be reviewed.

Review of plane and solid high school geometry for a one-semester remedial course.

Solid Geometry. By Mandelbaum and Conte (Wayne State University and Ramo-Wooldridge Corporation). Ronald Press, 1957. 261 pages, $4.50.

College Geometry. By Miller (Ohio State University). Appleton-Century-Crofts, 1957. 201 pages, $4.50. To be reviewed.

Circles (Volume II in the International Series of Monographs in Pure and Applied Mathematics). By Pedoe (University College, Khartoum, Sudan). Pergamon Press, 1957. 78 pages, $3.75. To be reviewed.

The 13 Books of Euclid's Elements (three volumes). Edited by Heath. Dover, 1956. Vol. I, 448 pages; Vol. II, 448 pages; Vol. III, 560 pages; $2.00 per volume.

The Real Projective Plane. By Coxeter (University of Toronto). Cambridge University Press, New York, 1955. 244 pages, $5.50.

An introductory university textbook in projective geometry which includes a thorough treatment of conics and a rigorous presentation of the synthetic approach to coordinates.

Famous Problems of Elementary Geometry, and other Monographs. By Klein, *et al.* (Goettingen, *etc.*). Chelsea, 1955. 350 pages, $3.25.

Fundamental Concepts of Geometry. By Meserve (New Jersey State Teachers College, Upper Montclair). Addison-Wesley, 1955. 321 pages, $7.50. Review: pp. 673–4, 1956.

Geometry of Four Dimensions. By Manning. Dover, 1954. 348 pages, $1.95. A revision of the 1914 volume.

STATISTICS AND RELATED TOPICS

Elementary Decision Theory. By Chernoff and Moses (Stanford University). Wiley, 1959. 376 pages, $6.25.

A modern introduction to statistics based on decision theory.

Sampling Inspection Tables. By Dodge and Romig (Rutgers University and Paper-Mate Mfg. Co.). Wiley, 1959. 224 pages, $7.00.

Quality Control and Industrial Statistics. Revised Edition. By Duncan (Johns Hopkins University). Richard D. Irwin, 1959. 979 pages, $9.00 text, $10.80 trade.

Information Theory and Statistics. By Kullback (George Washington University). Wiley, 1959. 344 pages, $12.50.

A study of information theory from the viewpoint of a general probabilistic meeting.

An Introduction to Multivariate Statistical Analysis. By Anderson (Columbia University). Wiley, 1958. 374 pages, $12.50. To be reviewed.

Presents basic methods used in multivariate statistical analysis—the analysis of everal variables.

Experimental Designs in Industry. By Chew (No. Carolina State College). Wiley, 1958. 268 pages, $6.00. To be reviewed.

A discussion of designs most applicable in industrial research.

An Introduction to Statistical Mechanics. By Chisholm and DeBorde (University College, Cardiff, and University of Glasgow). Pergamon Press, 1958. 400 pages, $6.00.

Introductory Statistics. By Quenouille. Pergamon Press. 247 pages, $6.00.

Manual of the more common statistical tests.

Planning of Experiments. By Cox (Birbeck College). Wiley, 1958. 308 pages, $6.25. To be reviewed.

An introductory treatment of particular interest to applied statisticians and experimental scientists.

Statistics—An Introduction. By Fraser (University of Toronto). Wiley, 1958. 44 pages, $7.25.

Modern Business Statistics. By Freund and Williams (Arizona State College and University of Tennessee). Prentice-Hall, 1958. 608 pages, $7.50. To be reviewed.

Experimental Design in Psychology and the Medical Sciences. By Maxwell (University of London). Wiley, 1958. 147 pages, $3.75. To be reviewed.

Deals with the relatively small number of basic, yet straightforward, experimental designs—randomized blocks, latin squares, and factorial designs—plus a few others neither so familiar nor so straightforward.

An Introduction to Combinatorial Analysis. By Riordan (Bell Telephone Lab.). Wiley, 1958. 244 pages, $8.50. To be reviewed.

Some Aspects of Multivariate Analysis. By Roy (University of No. Carolina). Wiley, 1958. 214 pages, $8.00. To be reviewed.

Sampling Opinions, An Analysis of Survey Procedure. By Stephan and McCarthy (Princeton University and Cornell University). Wiley, 1958. 451 pages, $12.00. To be reviewed.

Experimental Designs. By Cochran and Cox (Harvard University and University of North Carolina). Wiley, 1957. 617 pages, $8.50. Review: p. 756, 1957.

Nonparametric Methods in Statistics. By Fraser (University of Toronto). Wiley, 1957. 299 pages, $8.50. Review: p. 684–5, 1957.

An Introduction to Genetic Statistics. By Kempthorne (Iowa State College). Wiley, 1957. 545 pages, $12.75. Review: pp. 460–1, 1958.

Games and Decisions: Introduction and Critical Survey. By Luce and Raiffa (Harvard University). Wiley, 1957. 509 pages, $8.75. To be reviewed.

A survey of the central ideas and results of game theory and related decision making models. Not particularly mathematical.

The Essentials of Educational Statistics. By Cornell (Engelhardt and Leggett). Wiley, 1956. 375 pages, $5.75.

Statistical Analysis of Stationary Time Series. By Grenander and Rosenblatt (University of Stockholm and Indiana University). Wiley, 1956. 300 pages, $11.00. Review: pp. 218–9, 1958.

Government Statistics for Business Use. By Hauser and Leonard (University of Chicago). Wiley, 1956. 440 pages, $7.00.

Symposium on Monte Carlo Methods. By Meyer (University of Florida). Wiley, 1956. 382 pages, $7.50.

Papers by twenty-two leading workers in the field, written about their own research and applications.

Thermodynamics and Statistical Mechanics. By Wilson (Cambridge University). Cambridge University Press, New York, 1956. 500 pages, $9.50.

An Introduction to Stochastic Processes. By Bartlett (University of Manchester). Cambridge University Press, New York, 1955. 316 pages, $7.50. Review: p. 134–5, 1956.

Stochastic Models for Learning. By Bush and Mosteller (Columbia University and Harvard University). Wiley, 1955. 365 pages, $9.00.

The first systematic attempt to present a probabilistic analysis of data obtained in learning experiments through the use of stochastic processes.

Applied General Statistics, Second Edition. By Croxton and Cowden (Columbia Uni-

versity, University of North Carolina). Prentice-Hall, 1955. 843 pages, $6.70.

An elementary text with no college mathematics prerequisite.

Statistical Methods. By Mills (Columbia University). Holt, 1955. 864 pages, $7.95.

Statistical Analysis in Chemistry and the Chemical Industry. By Bennett and Franklin (General Electric Company and Leeds University). Wiley, 1954. 724 pages, $9.50.

Theory of Games and Statistical Decisions. By Blackwell and Girshick (University of California). Wiley, 1954. 355 pages, $7.75. Review: pp. 553–4, 1955.

Elements of Statistics. By Fryer (Kansas State College). Wiley, 1954. 262 pages, $4.75. Review: pp. 725–6, 1954.

Limit Distributions for Sums of Independent Random Variables. By Gnedenko and Kolmogorov (Moscow and Lwow Universities), translated from the Russian by Chung. Addison-Wesley, 1954. 264 pages, $12.00.

This volume provides the first appearance in English of much of its contents.

Introduction to Mathematical Statistics. By Hoel (University of California). Wiley, 1954. 331 pages, $5.00.

Mathematics of Statistics, Part I, Third Edition. By Kenney and Keeping (University of Wisconsin and University of Alberta). Van Nostrand, 1954. 335 pages, $5.25.

Business and Economic Statistics. By Spurr, Kellogg, and Smith (Stanford University, Deere and Company and The American University). Irwin, 1954. 590 pages, $6.95 text, and $8.35 trade.

Mathematical Foundations of Statistical Mechanics. By Khinchin. Dover. 1954, 179 pages, $1.35.

Decision Processes. By Thrall, Coombs and Davis (University of Michigan). Wiley, 1954. 332 pages, $5.00. Review: pp. 596–8, 1955.

Containing the material of 23 contributing scientists, this book offers an interdisciplinary approach to decision making under certainty.

The Foundations of Statistics. By Savage (University of Chicago). Wiley, 1954. 294 pages, $7.25.

Selected Papers on Noise and Stochastic Process. By Wax. Dover, 1954. 352 pages, $2.35.

Six basic papers for newcomers in the field of noise characteristics.

PROBABILITY

Probability: An Intermediate Textbook. By Bizley (Institute of Actuaries). Cambridge University Press, New York, 1957. 240 pages, $4.00.

An Introduction to Probability Theory and Its Applications, Volume I. By Feller (Princeton University). Wiley, 1957. 461 pages, $9.00. Review: p. 538, 1958.

Treats probability theory rigorously as a self-contained mathematical subject, and demonstrates how practical problems may be solved through application of this theory.

Elements of Probability Theory and Some of Its Applications. By Cramer (University of Stockholm). Wiley, 1955. 281 pages, $7.00. Review: pp. 132–4, 1956.

A compact discussion of the mathematical theory of probability theory with emphasis on random variables and probability distributions.

Probability Theory. By Loeve (University of California). Van Nostrand, 1955. 532 pages, $12.75.

A text for students having a knowledge of classical analysis.

TOPOLOGY AND ALGEBRAIC GEOMETRY

Convex Surfaces. By Busemann (University of Southern California). Interscience, 1958. 204 pages, $6.00.

Convexity. By Eggleston (Cambridge University). Cambridge University Press, New York, 1958. 160 pages, $4.00. To be reviewed.

Introduction to Algebraic Geometry. By Lang (Columbia University). Interscience, 1958. 272 pages, $7.25. To be reviewed.

Homology Theory on Algebraic Varieties (*Volume 6 in the International Series of Monographs in Pure and Applied Mathematics*). By Wallace (University of Toronto). Pergamon Press, 1958. 115 pages, $5.50. To be reviewed.

Topological Analysis. By Whyburn (University of Virginia). Princeton University Press, 1958. 124 pages, $4.00. To be reviewed.

Geometric Algebra. By Artin (Princeton University). Interscience, 1957. 224 pages, $7.00. Review: pp. 604–5, 1957.

This text discusses the foundations of affine geometry, and geometry of quadratic forms and the structure of the general linear group. It also contains a discussion of projective and symplectic and orthogonal groups.

Lie Groups. By Cohn. (University of Manchester). Cambridge University Press, 1957. 178 pages, $4.00. Review: p. 646, 1958.

Problems in Euclidean Space—Application of Convexity (*Volume 5 in the International Series of Monographs in Pure and Applied Mathematics*). By Eggleston (Cambridge University). Pergamon Press, 1957. 165 pages, $6.50. To be reviewed.

Algebraic Geometry and Topology. Fox, Spencer, Tucker (Princeton University). Princeton University Press, 1957. 410 pages, $7.50. To be reviewed.

Introduction to Riemann Surfaces. By Springer (University of Kansas). Addison-Wesley, 1957. 307 pages, $9.50. To be reviewed.

An Introduction to Algebraic Topology (*Volume I in the International Series of Monographs in Pure and Applied Mathematics*). By Wallace (University of Toronto). Pergamon Press, 1957. 198 pages, $6.50. Review: pp. 466–7, 1958.

The Theory of Lie Derivatives and Its Applications. By Yano (University of Tokyo). Interscience, 1957. 303 pages, $9.00. Review: pp. 294–5, 1958.

Topology. By Patterson (University of St. Andrews). Interscience, 1956. 136 pages, $1.75.

Topological space is approached gradually through Euclidean and metric spaces. The separation axioms, compactness, and connectedness, are then discussed, and also homotopy, with an elementary account of the homology theory of simplicial complexes, as a brief introduction to algebraic topology.

Elementary Topology. By Hall and Spencer (Harpur College and Williams College). Wiley, 1955. 303 pages, $7.00. Review: pp. 591–2, 1956.

General Topology. By Kelley (University of California). Van Nostrand, 1955. 298 pages, $8.75. Review: pp. 668–9, 1955.

Topological Transformation Groups. By Montgomery and Zippin (Princeton University and Queens College). Interscience, 1955. 294 pages, $6.50. Review: pp. 439–40, 1956.

LOGIC AND SET THEORY

Introduction to the Theory of Sets. By Breuer and translated by Fehr (Aachen, Germany; Columbia University Teachers College). Prentice-Hall Inc., 1958. 108 pages, $4.25. To be reviewed.

Introduction to Logic and Sets, Preliminary Edition. By Christian (University of British Columbia). Ginn, 1958. 70 pages, $0.90. To be reviewed.

A Modern Introduction to Logic. By Blyth (Hamilton College). Houghton Mifflin, 1957. 426 pages, $5.50.

Set Theory. By Hausdorff (Greifswald University). Chelsea, 1957. 352 pages, $6.00.

Introduction to Logic. By Suppes (Stanford University). Van Nostrand, 1957. 336 pages, $5.50. Review: pp. 131–2, 1958.

101 Puzzles in Thought and Logic. By Wylie. Dover, 1957. 128 pages, $1.00. Review: p. 648, 1958.

Introduction to Mathematical Logic. By Church (Princeton University). Princeton University Press, 1956. 400 pages, $7.50. Review: pp. 126–7, 1957.

Theory of Sets. By Kamke. Dover, 1956. 152 pages, $1.35.

The Elements of Mathematical Logic. By Rosenbloom (University of Minnesota). Dover, 1955. 214 pages, $1.45.

An Introduction to Symbolic Logic. By Langer. Dover, 1954. 368 pages. $1.75.

TEACHING

Arithmetic for Colleges. By Larsen (Albion College). Macmillan, 1958. 286 pages, $5.50. To be reviewed.

History of Mathematics, Volume I. By Hoffmann (Oklahoma State University). Philosophical Library, 1957. 136 pages, $4.75. Review: p. 692, 1957.

Understanding Arithmetic. By Swain (State University of New York, New Paltz). Rinehart, 1957. 416 pages, $4.75. Review: pp. 296–7, 1958.

An Introduction to Mathematics, Revised Edition. By Boyer (College Preparatory, Department of Public Instruction, Harrisburg, Pa.). Henry Holt, 1956. 528 pages. $5.75.

Teaching of Mathematics. Cambridge University Press, 1956. 275 pages, $3.00.

Arithmetic: Its Structure and Concepts. By Mueller (Towson State, Md., Teachers College). Prentice-Hall, 1956. 279 pages, $6.25.

Arithmetic for Teacher-Training Courses. By Mills and Taylor (Sioux Falls College and Eastern Illinois State College). Holt, 1955. 438 pages, $4.85.

Elementary Mathematics from an Advanced Standpoint: Volume I—Arithmetic, Algebra, Analysis. By Klein. Dover, 1954. 214 pages, $1.75.

Elementary Mathematics from an Advanced Standpoint: Volume II—Geometry. By Klein. Dover, 1955. 214 pages, $1.75.

A reissue of Klein's older book.

Mathematics for the Secondary School. By Reeve (Columbia University), Holt, 1954. 535 pages, $6.95. Review: p. 739, 1955.

The Teaching of Arithmetic, Second Edition. By Spitzer (University of Iowa). Houghton Mifflin, 1954. 416 pages, $4.25.

UNCLASSIFIED

Mathematics in Fun and in Earnest. By Court (University of Oklahoma). Dial Press, 1958. 250 pages, $4.75. To be reviewed.

A delightful "popular book" by one of today's great synthetic geometers.

How to Study, How to Solve (*Special Student Edition*), Second Edition. By Dadourian (Trinity College). Addison-Wesley, 1958. 48 pages, $0.50.

An Introduction to the Foundations and Fundamental Concepts of Mathematics. By Eves and Newsom (University of Maine and New York University). Rinehart, 1958. 363 pages, $6.75. Review: p. 720, 1958.

Introduction to Mathematical Economics. By Bushaw and Clower (State College of Washington and Northwestern University). Richard D. Irwin, 1957. 357 pages, $7.00 text, $8.40 trade.

The Tree of Mathematics. Edited by James (Editor, Mathematics Magazine). Digest Press, 1957. 402 pages, $5.50. To be reviewed.

A series of articles from the *Mathematics Magazine.*

An Introduction to the Mechanics of Stellar Systems. By Kurth, translated by Kahn (University of Manchester). Pergamon Press, 1957. 175 pages, $9.00.

The Enjoyment of Mathematics. By Rademacher and Toeplitz (University of Pennsylvania). Princeton University Press, 1957. 240 pages, $4.50. Review: p. 603, 1957.

Mathematical Vocabulary (German-English). By Macintyre and Witte (Aberdeen University). Interscience, 1956. 106 pages, $1.75.

Through the Mathescope. By Ogilvy (Hamilton College). Oxford University Press, 1956. 162 pages, $4.00. Review: p. 592–3, 1956.

Prelude to Mathematics. By Sawyer (St. John's College, Cambridge). Penguin Books 1955. 224 pages, $0.85.

Monographs on Topics of Modern Mathematics. By Young. Dover, 1955. 416 pages, $2.00.

Relativity for the Layman. By Coleman (American International College, Springfield, Mass). William-Frederick Press, 1954. 224 pages, $2.75.

Mathematics and Plausible Reasoning: Volume I—Induction and Analogy in Mathematics. By Polya (Stanford University). Princeton University Press, 1954. 300 pages, $5.50 ($9.00 per set). Review: pp. 456–7, 1958.

Mathematics and Plausible Reasoning: Volume II—Patterns of Plausible Inference. By Polya (Stanford University). Princeton University Press, 1954. 200 pages, $4.50 ($9.00 per set). Review: pp. 456–7, 1958.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1959.

Columbia University, July 6 to August 14: Mr. Thorpe, introduction to higher algebra; Mr. Glimm, differential equations; Mr. Clifton, probability; Professor Dorothy M. Stone, fundamental concepts of mathematics; Professor Taylor, general topology; Dr. Mendelson, theory of functions of a real variable; Professor A. H. Stone, theory of functions, third part; Dr. Taft, higher algebra, second part.

Columbia University, Teachers College, July 6 to August 14: Miss Allegri, teaching of geometry and teaching of junior high school mathematics; Dr. Rosskopf, teaching modern mathematics in the secondary school and foundations of geometry; Dr. Greitzer, applications of mathematics and professionalized subject matter in advanced secondary school mathematics.

Duke University, June 12 to July 17: Professor Roberts, advanced calculus; Professor Gergen, mathematical statistics.

Kent State University, June 22 to July 25; Professor Jenkins, theory of equations; Professor Kaiser, probability; Professor Stapleford, advanced methods of teaching high school mathematics. July 27 to August 29: Professor Iwanchuk, selected topics for classroom teachers, functions of a real variable; Professor Olson, differential geometry.

New York University, first term: Dr. Bazer, Dr. Schechter, advanced calculus (two sessions); Professor Isaacson, elementary numerical methods; Professor Shapiro, special functions. Second term: Dr. Kay, Mr. Ungar, advanced calculus (two sessions); Professor Karp, Laplace transform and Heaviside calculus.

Northwestern University, six week session June 23 to August 1, eight week session June 23 to August 15: advanced calculus, engineering mathematics 1-T; numerical methods in mathematics; introduction to the theory of numbers; the history of mathematics II; advanced geometry for teachers; foundations of calculus for teachers; complex variables for applications; topics in modern mathematics for teachers.

Syracuse University, June 29 to August 7: Professor Cole, analysis and applications; Professor Coté, statistics and probability; Professor Gilbert, modern algebra; Professor Hemmingsen, projective geometry; Staff, number theory, programming digital computers. Syracuse University will also offer two demonstration classes in new high school curricula: an eighth-grade class in the Beberman-Illinois-UICSM "First Course," and an eleventh-twelfth grade class in statistics, following the text prepared by the CEEB Commission on Mathematics. Anyone interested is invited to visit these courses at any time between June 30 and August 7.

University of Chicago, June 22 to August 29: The program stresses homological algebra. Advanced courses: Professor Hochschild, topics in Lie algebras; Professor MacLane, foundations of homological algebra; Professor Baily, analytic number theory; Professor Calderon, potential theory; Professor Halmos, ergodic theory; Professor Spanier, characteristic classes. Standard graduate courses: Professor Halmos, set theory and metric spaces; Professor Kaplansky, algebra IV; Dr. Swan, algebraic topology II.

University of Florida, June 16 to August 8: Professor Blake, Professor Meacham, introduction to mathematical thought; Professor South, mathematical statistics, advanced

topics in calculus; Professor Butson, Professor Gaddum, Professor Meacham, advanced mathematics for engineers and physicists; Professor Moore, introduction to topology; Professor Pirenian, vector analysis, theory of groups of finite order; Professor Smith, tensor analysis; Professor Morse, synthetic projective geometry, history of elementary mathematics; Professor Phipps, foundation of geometry; Professor Cowan, Fourier series; Professor Sobczyk, special topics in mathematics.

University of Maryland, June 22 to July 31: Professor Horváth, advanced calculus, higher geometry; Professor Rosen, vector analysis.

University of Michigan, June 22 to August 15: Professor Addison, introduction to the foundations of mathematics; Dr. Albrecht, differential equations, advanced mathematics for engineers; Professor Bartels, methods in high-speed computation (computer algorithms); Professor Carver, calculus of finite differences; Professor Clarke, mathematical theory of probability, theory of statistics I; Professor Coburn, vector analysis; Professor Craig, statistical analysis II; Professor Griffin, advanced calculus; Dr. Halpern, introduction to combinatorial topology; Professor Hay, intermediate course in differential equations; Dr. Hicks, differential geometry; Professor Jones, the teaching of elementary collegiate mathematics; Dr. Karrer, advanced mathematics for engineers, Fourier series and applications; Dr. Kincaid, statistical analysis I, theory of statistics II; Professor Kruskal, differential equations, introduction to matrices; Professor Leisenring, synthetic projective geometry; Professor Livingstone, theory of equations and determinants; Dr. Low, modern operational mathematics; Professor Lyndon, introduction to functions of a complex variable with applications, algebra; Professor Mayerson, differential equations; Professor McLaughlin, modern operational mathematics, topics in modern mathematics for teachers; Professor Reade, introduction to functions of a complex variable with applications, real analysis; Professor Ullman, differential equations, Fourier series and applications.

University of Minnesota, College of Science, Literature, and the Arts, June 15 to July 18: Professor Gil de Lamadrid, differential equations, theory of numbers; Professor Orey, critical reasoning in mathematical analysis. July 20 to August 22: Professor Engeler, advanced analytic geometry, advanced algebraic theory; Professor Storvick, calculus of variations.

University of Minnesota, Institute of Technology, June 15 to July 18: Professor Wilcox, vector analysis; Professor Munro, introduction to programming modern digital calculators, elementary numerical analysis in engineering; Professor Polansky, intermediate calculus. July 20 to August 21: Professor Wilcox, advanced calculus.

University of Nebraska: Professor Leavitt, theory of equations; Professor Harris, theory of games; Professor Miller, topics in algebra; Professor Guy, topics in analysis.

University of North Carolina, June 4 to July 14: Professor Cameron, fundamental concepts of mathematics with special reference to algebra; Professor Davis (University of Virginia), probability; Professor Garner, history of mathematics; Professor Hill, elementary mathematical statistics; Professor Linker, differential equations; Professor Mac Nerney, introduction to Hilbert spaces; Professor Pettis, topics in analysis; Mr. Wells, advanced calculus I, summability. July 15 to August 22: Professor Brauer, elementary theory of numbers I; Professor Hamstrom (Goucher College), elementary algebraic topology; Professor Hoyle, advanced calculus II; Professor Lasley, analytic geometry from a higher standpoint; Professor Mackie, theory of equations; Professor Mac Nerney, introduction to continued fractions; Professor Mann, introduction into numerical analysis.

University of Oklahoma, June 9 to August 6: Professor Bernhart, college geometry, vector analysis; Professor Pan, elementary differential equations; Professor LaFon, partial differential equations; Professor Brixey, theory of groups; Professor Giever, advanced partial differential equations.

University of Pittsburgh, June 8 to July 17, July 20 to August 28: Professors Christiano and Knipp, differential equations; Professor Benedicty, advanced calculus; Professor Taylor, functions of a complex variable; Professor Bryson, partial differential equations and Fourier series; Professor Bompiani, projective differential geometry; Professor Laush, infinite series; Professor Levine, topology. June 23 to August 15: Professor Elyash, differential equations; Professor Kovacs, mathematical theory of statistics; Professor Teats, history of mathematics; Professor Myers, recreational mathematics for teachers, theory of equations; Professor Leger, matrix theory. June 22 to August 14 (evenings): Professors Leger and Levine, differential equations; Professors Blumberg and Laird, mathematics of modern engineering; Professor Cooperman, advanced calculus; Professor Bryson, Laplace transform theory and applications.

University of Virginia, June 29 to August 22: Professor Ball, foundations of geometry; Professor Paige, foundations of algebra; Professor Malbon, differential equations and applied mathematics; Professor Paige, introductory analysis; Visiting Professor, advanced analysis.

University of Washington; Professor Walter, linear algebra; Professor Avann, introduction to modern algebra; Professor Brownell, topics in applied analysis; Professor Michael, advanced analytic and non-Euclidean geometry; Professor Pierce, foundations of mathematics; Professor Allendoerfer, special topics in modern mathematics for teachers.

University of Wisconsin, June 22 to August 15: Mr. Albright, theory and operation of computing machines; Professor Bicknell, applied differential equations; Visiting Lecturer Goblirsch, survey of the foundation of algebra; Visiting Lecturer Immel, applied mathematical analysis, introduction to mathematical probability; Dean Ingraham, topics in classical algebra; Dr. Losey, advanced analytic geometry; Visiting Lecturer Montgomery, advanced topics in algebraic topology; Dr. Posner, introduction to complex variables; Visiting Lecturer Sanderson, advanced calculus, elementary plane topology; Visiting Lecturer Schenkman, matrices and their applications, advanced topics in algebra.

West Virginia University, June 8 to July 17: Miss Hawkins, differential equations; Mrs. Easton, theory of equations; Professor Cunningham, group theory; Professor Posey, calculus of variations. July 20 to August 26: Mr. Gould, number theory; Professor Cochran, special topics; Professor Vest, advanced differential equations; Professor Peters, linear algebra.

OPERATIONS RESEARCH SOCIETY OF AMERICA

The Seventh Annual Meeting of the Operations Research Society of America will be held at the Shoreham Hotel, Washington, D. C., May 14–15, 1959.

PERSONAL ITEMS

Miss Mabel Williams of Tyler Junior College represented the Association at the inauguration of President R. W. Steen of Stephen F. Austin State College, Nacogdoches, Texas, on February 7, 1959.

University of North Carolina: Professor N. H. McCoy, Smith College, will be a Visiting Professor for the fall semester of 1959–1960; Professor T. H. Hildebrandt, University of Michigan, will be a Visiting Professor for the spring semester of 1959–1960.

Mathematics Research Center, U. S. Army, University of Wisconsin: Professor J. G. van der Corput, Mathematics Center, Amsterdam, Netherlands, Dr. H. M. Lieberstein, Ramo-Wooldridge Corporation, and Dr. S. M. Shah of India, have accepted appointments; Dr. Eberhard Hopf, Indiana University, has accepted a visiting appointment.

Professor Volodymyr Bohun-Chudyniv, Atlanta University, has been appointed Professor at Morgan State College.

Mr. K. C. Bullock, Oklahoma State University, has been appointed Instructor at

Murray State College.

Mr. W. H. Burgin, Jr., Princeton University, has been appointed Instructor at the Mercersburg Academy, Mercersburg, Pennsylvania.

Mr. R. L. Causey, Space Technology Laboratories, Los Angeles, California, has accepted a position as Senior Scientist with Lockheed Missiles Systems Division, Palo Alto, California.

Professor M. L. Curtis, University of Georgia, has received a Senior Postdoctoral Fellowship and will spend the year 1959–1960 in England.

Mr. C. M. Fast, General Electric Company, Syracuse, New York, has accepted a position as Mathematician with the Kirk Engineering Company, Philadelphia, Pennsylvania.

Mr. S. I. Gass, International Business Machines Corporation, has been appointed Chief, Operations Research Branch, Corporation for Economic and Industrial Research, Arlington, Virginia.

Dr. J. H. Griesmer, International Business Machines Corporation, Ossining, New York, has been appointed Staff Mathematician at the I.B.M. Research Center, Yorktown Heights, New York.

Professor G. B. Huff, University of Georgia, has resigned as Head of the Department of Mathematics to become Dean of the Graduate School.

Mr. M. R. Luttrell, Lincoln Laboratories, Massachusetts Institute of Technology, Lexington, Massachusetts, has been appointed Field Engineer, Sage Air Defense System of the Western Electric Company, Inc., New York.

Assistant Professor J. H. Manheim, Cooper Union, New York, has been appointed Assistant Professor at Montclair State College.

Assistant Professor Rev. P. M. Mino, College of Steubenville, has been appointed Assistant Professor at St. Francis College.

Assistant Professor V. J. Mizell, University of Tennessee, has been appointed Assistant Professor at Carnegie Institute of Technology.

Mr. P. M. Moskowitz, Metropolitan Life Insurance Company, New York City, is now a Senior Systems Analyst with the Radio Corporation of America, Princeton, New Jersey.

Mrs. Doretta M. Pasekoff, Ursinus College, has accepted a position as Mathematician in Univac Methods Department, Remington Rand Corporation, Pittsburgh, Pennsylvania.

Mr. B. L. Schwartz, Battelle Memorial Institute, has accepted a position as Mathematician with Technical Operations, Inc., Monterey, California.

Mr. D. J. Sine, West Virginia University, has accepted a position as Mathematician with Aero-Jet-General, Frederick, Maryland.

Sister Mary Vera, B.V.M., Mundelein College, has been appointed Chairman of the Mathematics Department at Clarke College.

Assistant Professor R. G. Stoneham, Santa Barbara College, has been appointed Teacher of Mathematics at The Walden School, New York.

Mr. D. V. Sward, Montana State University, has accepted the position of Associate Engineer with the Boeing Airplane Company, Seattle, Washington.

Assistant Professor Peter Yff, Fresno State College, has been appointed Associate Professor at the American University of Beirut, Beirut, Lebanon.

Professor Emeritus J. N. Michie, Texas Technological College, died on November 24, 1958. He was a member of the Association for 33 years.

Professor G. H. Pavlakos, Elmhurst College, died on November 10, 1958. He was a member of the Association since 1956.

Professor Emeritus B. F. Yanney, College of Wooster, died on August 10, 1958. He was almost 99 years old, and was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-SECOND ANNUAL MEETING OF THE ASSOCIATION

The forty-second annual meeting of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pennsylvania on Thursday and Friday, January 22 and 23, 1959, in conjunction with meetings of the American Mathematical Society, the Association for Symbolic Logic, and the Delaware Valley Section of the Society for Industrial and Applied Mathematics. There were registered 996 persons, including 621 members of the Association.

Sessions of the Association were held on Thursday morning and on Friday morning and afternoon in the Auditorium of the University Museum of the University of Pennsylvania. President G. B. Price presided at the sessions on Thursday and Friday mornings and at the Annual Business Meeting. Vice-President B. W. Jones presided at the Friday afternoon session. The Program Committee for the meeting consisted of J. G. Kemeny, Chairman; E. G. Begle, and M. A. Shader.

FIRST SESSION OF THE ASSOCIATION

"Professional Opportunities in Mathematics." Moderator: Dr. M. A. Shader, IBM Corporation; Panel Members: Professor Wallace Givens, Wayne State University, Dr. H. H. Goldstine, IBM Corporation, Dr. Arthur Grad, National Science Foundation, Dr. J. P. Nash, Lockheed Aircraft Corporation.

SECOND SESSION OF THE ASSOCIATION

"The Training of Secondary School Mathematics Teachers." Moderator: Professor J. G. Kemeny, Dartmouth College; Panel Members: Professor C. B. Allendoerfer, University of Washington, Professor H. F. Fehr, Teachers College, Columbia University, Professor E. P. Northrop, University of Chicago, Professor H. E. Vaughan, University of Illinois.

THIRD SESSION OF THE ASSOCIATION

Annual Business Meeting of the Association.

"High School Mathematics Courses." (Co-sponsored by the School Mathematics Study Group.) Moderator: Professor E. G. Begle, Yale University and School Mathematics Study Group; Panel Members: Dr. J. R. Mayor, American Association for the Advancement of Science, Professor E. E. Moise, University of Michigan, Dr. H. O. Pollak, Bell Telephone Laboratories, Professor G. B. Price, University of Kansas, Mr. R. E. K. Rourke, Kent School and Commission on Mathematics, Professor D. E. Richmond, Williams College.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Thursday afternoon in the Jefferson Room of the Benjamin Franklin Hotel in Philadelphia with twenty-three members present. Among the items of business transacted were the following:

The Board approved the appointment by President Price of the following Nominating Committee for 1958: Mina Rees, Chairman, E. G. Begle, and A. E. Taylor.

The Board elected Professor Harley Flanders of the University of California as Second Vice-President for the two-year term 1959–1960.

Since the Board was not prepared to elect a Secretary, Professor H. M. Gehman was instructed to continue to act as Secretary of the Association.

The Board approved the following schedule of future meetings: University of Utah, August 31–September 3, 1959; Hotel Conrad Hilton, Chicago, January 28–30, 1960; Michigan State University, August 29–September 1, 1960; Washington, D. C., or New York City, January 1961; Oklahoma State University, August 28–31, 1961; Kansas City, Missouri, January 1962.

The Board voted to invite Professor William Feller of Princeton University to deliver the eighth Earle Raymond Hedrick Lectures at the 1959 Summer Meeting.

The Board authorized the publication as Carus Monograph No. 12 of "Statistical Independence in Probability, Analysis, and Number Theory," by Mark Kac.

The Board voted to accept with an expression of gratitude a grant from the Carnegie Corporation of New York of $75,000 over a three year period for the support of a Washington Office of the Association.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Friday, January 23, 1959 in the Auditorium of the University Museum of the University of Pennsylvania, Philadelphia, Pennsylvania. President G. B. Price presided.

The ten amendments to the By-Laws which were printed in the November 1958 issue of the MONTHLY were unanimously adopted.

The balloting for officers in which 1046 votes were cast resulted in the election of Professor C. B. Allendoerfer of the University of Washington as President for the two-year term 1959–1960, and of Professors R. V. Churchill of the University of Michigan and Morris Kline of New York University as governors for the three-year term 1959–1961.

In spite of a more stringent policy on dropping from membership for non-payment of dues, membership in the Association was 7754 on January 16, 1959.

Reports were made by Dr. J. R. Mayor for the Committee on Secondary School Lecturers and by Professor L. W. Cohen for the Committee on Production of Films.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Tuesday, January 20, through Thursday. The annual Gibbs lecture was delivered by Professor J. M. Burgers of the University of Maryland. Invited addresses were given by Professors G. D. Mostow and Felix Browder.

The Association for Symbolic Logic met on Thursday and the Delaware Valley Section of SIAM met on Wednesday evening for a lecture by Dr. John Mauchly and a social get-together.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: Emil Grosswald, Chairman, P. A. Caris, J. H. Curtiss, Robert Ellis, H. M. Gehman, W. H. Gottschalk, R. D. Schafer, G. E. Schweigert, C. T. Yang.

Registration headquarters were in the foyer of Houston Hall of the University of Pennsylvania. The book exhibit and the employment register were located on the second floor of Houston Hall. Sleeping accommodations were available in the Benjamin Franklin Hotel and in other hotels. Meals were available at the hotels and at several University cafeterias.

An official reception by the University of Pennsylvania was held in the Rotunda of the University Museum on Wednesday afternoon.

A banquet was held on Thursday evening at the Benjamin Franklin Hotel. Professor Bernard Epstein served as Chairman. Professor J. S. Frame presented a resolution on behalf of the mathematical organizations expressing thanks to the administration of the

University of Pennsylvania, to the members of the Mathematics Department, and particularly to the members of the local committee on arrangements for their successful efforts on our behalf.

HARRY M. GEHMAN, *Secretary Treasurer*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The Philadelphia Section of the Mathematical Association of America met at Lehigh University on Saturday, November 29, 1958, with Professor Charles Saalfrank presiding in lieu of the Section Chairman, Dr. I. E. Block. There were fifty-four present at this meeting, of whom thirty-nine are members of the Association.

For the year 1958–59 Professor Marguerite Lehr, Bryn Mawr College, was elected Chairman of the Section, Professor F. L. Dennis, Ursinus College, was elected Secretary-Treasurer, and Professor Bernard Epstein, University of Pennsylvania, was elected to the Executive Committee.

The following resolution was adopted by the Section:

Be it resolved that the Philadelphia Section of the Mathematical Association of America direct and empower the Executive Committee to take the necessary steps to institute a Mathematics Newsletter directed primarily toward the secondary schools of the area included in the Section or to take other action directed toward this goal if such is deemed desirable.

The entire meeting was devoted to the topic "Desirable Mathematical Training for the Mathematician Who Plans to Work in Industry." This was discussed in the morning program in three papers, as follows:

1. Dr. R. F. Drenick, Bell Telephone Laboratories. The mathematician's primary problem is communication, especially with nonmathematical colleagues. The gap between pure and applied mathematicians must be closed. Synthesis is the mathematician's primary job.

2. Dr. H. D. Mills, Princeton University. The mathematician must have a basic empathy for other people's problems, solid mathematical scholarship, inventive capacity, and self-esteem as a mathematician. Students should receive a concept of "wholeness" of mathematics, a survey based on the "storehouse of mathematics" so he will know what is in the literature, and case histories of mathematical discovery.

3. Dr. H. R. J. Grosch, International Business Machines Corporation. The influence of mathematicians in industry will diminish unless the curriculum is broadened; the present curriculum was based on training necessary for engineers. Work in information theory, logic, combinatorial analysis, probability and statistics is needed.

The afternoon program consisted of a panel discussion. Members of the panel were: Dean Mina Rees, Hunter College, Moderator; Professor Everett Pitcher, Lehigh University, and the morning speakers.

G. C. WEBBER, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Fall Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held on December 6, 1958, at George Washington University, Washington, D. C. Professor Joseph Milkman of the U. S. Naval Academy and Professor Herta T. Freitag of Hollins College presided. Mr. W. H. Norris, Section Chairman of the Mathematics Contest, announced the plans for the 1959 Contest.

The following papers were presented at the meeting:

1. *A mathematical model for transfer of training*, by Professor J. M. Long, College of William and Mary in Norfolk.

In this paper the author describes a mathematical model or analogy for the study of the transfer of training problem. The idea is to get a model which when analyzed will give results similar to that of a laboratory experiment. The "coefficient of transfer" is described and proposed as an appropriate quantitative measure of transfer. A simple geometrical model is the first model considered. This is expanded into an n-dimensional system with weighted parameters as the final model proposed.

2. *Algebraic compilers for scientific computers*, by Mr. R. C. Smith, Applied Science Representative, International Business Machines Corporation, introduced by the Secretary.

A progress report on the adaptation of computer language to mathematical language. A program in FORTRANSIT and the Soap Interpretive System for the solution of a mathematical problem on a computer.

3. *On the order of contact of two curves*, by Professor S. B. Jackson, University of Maryland.

Let two tangent curves have their points in 1-1 correspondence in the neighborhood of the contact point. The order of contact is defined by considering the distance d between corresponding points for a correspondence in which d is an infinitesimal of maximum possible order. A discussion is given of conditions under which a given correspondence is known to be optimal, and this result is applied to the case when corresponding points have equal arc lengths. It is shown that in general this is an optimal correspondence but an example is given to show this is not always the case.

4. *On the summability of a certain eigenfunction series*, by Professor Luna Mishoe, Morgan State College.

In this paper it is shown that if $u_n(x)$ are eigenfunctions or nonzero solutions of the equation $u''+q(x)+\lambda[P(x)u-u']=0$ such that $u(0)=u(1)=0$ and $P^2(x)+q(x)=0$ then the series: $\sum_{-\infty}^{\infty} a_n u_n(x)$ (where a_n are the corresponding eigen-coefficients determined in the usual manner) behaves exactly as the Fourier Series of $f(x)$, with respect to summability.

5. *Practical adaptations of Ferrari's general quartic solution*, by Mr. C. R. White, Special Problems Section, Artillery Weapon Systems Branch, Aberdeen Proving Ground, Maryland.

The fourth degree equation is resolved into two quadratic factors by the method of Ferrari, in which the coefficients of these factors are themselves the real roots of two other quadratic equations and are also functions of K, a Ferrari real root of Euler's reducing cubic. The author's contribution, in addition to the discovery of the two associated quadratic equations is to provide direct methods for evaluating K. The formulas which result are then transformed for computing purposes on machines designed to extract square roots directly such as a Friden calculator, a digital computer, or White's two-dimensional slide rule. Furthermore, the formulas provide the basis for an improved technique of completely solving the general nth degree equation with real coefficients.

6. *Weak limit characterization of distributions*, by Professor T. P. Liverman, George Washington University.

Let K be the set of piecewise continuous functions on E_1 to E_1. A test function $\phi(x)$ is a function on E_1 to E_1 of class C^∞ which vanishes identically outside a finite interval. The A. L. Schwartz distribution T is an equivalence class of sequences $\{t_n(x)\}$ of K-functions which are weakly convergent, *i.e.*, such that $I(\phi)=\lim_{n\to\infty}\int_{-\infty}^{\infty} t_n\phi(x)dx$ exists for every test function. With this approach, initiated by Mikusinski, G. Temple, and Lighthill, distributions, the natural tool in much applied

mathematics, should be accessible to undergraduates. This paper offers a simplification of the previously known theory; an elementary, though rigorous, justification of a commutative Heaviside calculus; and a proof, in elementary terms, of the fundamental theorem of distribution theory, *i.e.*, locally every distribution is a derivative of finite order of a continuous function.

7. *A new characterization of group*, by Professor Howard Campaigne, American University.

New investigations into semigroups conducted by G. B. Preston, V. V. Vagner, A. H. Clifford and others have led to the concept of "inverse" semigroups, in which for each element a there is a unique solution x to the simultaneous statements 1) $axa=a$ and $xax=x$. A solution to 1) implies a simultaneous solution of both, which suggests investigating semigroups which have unique solutions to 1). These are groups.

8. *Tetrahedra equivalent to cubes by dissection*, by Mr. Michael Goldberg, Bureau of Ordnance, U. S. Navy.

The list of special tetrahedra which can be dissected by plane cuts into a finite number of pieces to form cubes, published by J. J. M. Hill in 1896 and augmented by four new tetrahedra by J. P. Sydler in 1956, is now augmented by two new tetrahedra. Mr. Goldberg exhibited models of all the known dissectible tetrahedra and showed how the recent ones were derived. A tabulation of the lengths of the edges and dihedral angles was presented.

9. *Some published USSR research in mathematical economics*, by Dr. W. H. Marlow, Senior Staff Scientist, George Washington University. (By invitation.)

An exposition is given of some problems attacked by L. V. Kantorovich and various collaborators and by G. Sh. Rubinshtein in areas of organizing and planning production. These involve programming problems for which "Lagrange Multiplier" or "razreshaiushchiimnozhitel (resolving multiplier)" techniques have been devised by Kantorovich over the past twenty years. (Cf. *Doklady Akademii Nauk SSSR*, vol. 113 (1957), pp. 987–990; vol. 115 (1957, pp. 414–444; ibid., pp. 1058–1061; and errata: vol. 118 (1958), p. 1054.)

D. B. LLOYD, *Secretary*

OFFICERS AND COMMITTEES AS OF FEBRUARY 1, 1959

OFFICERS

President, C. B. ALLENDOERFER, University of Washington (1959–1960)
First Vice-President, G. B. THOMAS, JR., Massachusetts Institute of Technology (1958–1959)
Second Vice-President, HARLEY FLANDERS, University of California (1959–1960)
Editor, R. D. JAMES, University of British Columbia (1957–1961)
Secretary, H. M. GEHMAN, University of Buffalo (Acting)
Treasurer, H. M. GEHMAN, University of Buffalo (1958–1962)
Associate Secretary, L. J. MONTZINGO, JR., University of Buffalo (1958–1962)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

E. J. MCSHANE, University of Virginia (1955–1960)
W. L. DUREN, JR., University of Virginia (1957–1962)
G. B. PRICE, University of Kansas (1959–1964)

Governors at Large

H. M. BACON, Stanford University (1957–1959)
J. R. MAYOR, AAAS (1957–1959)

HOWARD EVES, University of Maine (1958–1960)
J. S. FRAME, Michigan State University (1958–1960)
R. V. CHURCHILL, University of Michigan (1959–1961)
MORRIS KLINE, New York University (1959–1961)

Sectional Governors (July 1, 1956–June 30, 1959)

Illinois, E. C. KIEFER, Millikin University
Iowa, BERNARD VINOGRADE, Iowa State College
Louisiana-Mississippi, Z. L. LOFLIN, Southwestern Louisiana Institute
Maryland-Dist. of Col.-Virginia, O. J. RAMLER, Catholic University of America
Michigan, B. M. STEWART, Michigan State University
Minnesota, G. K. KALISCH, University of Minnesota
Philadelphia, N. J. FINE, University of Pennsylvania
Southern California, P. H. DAUS, University of California, Los Angeles
Texas, C. R. SHERER, Texas Christian University

Sectional Governors (July 1, 1957–June 30, 1960)

Allegheny Mountain, J. C. KNIPP, University of Pittsburgh
Indiana, LAMBERTO CESARI, Purdue University
Kentucky, R. S. PARK, Eastern Kentucky State College
Metropolitan New York, JEWELL H. BUSHEY, Hunter College
Nebraska, W. G. LEAVITT, University of Nebraska
Northern California, GEORGE POLYA, Stanford University
Oklahoma, W. N. HUFF, University of Oklahoma
Rocky Mountain, C. R. WYLIE, JR., University of Utah
Wisconsin, H. P. EVANS, University of Wisconsin

Sectional Governors (July 1, 1958–June 30, 1961)

Kansas, R. G. SMITH, Kansas State College, Pittsburg
Missouri, W. R. UTZ, JR., University of Missouri
New Jersey, WILLIAM FELLER, Princeton University
Northeastern, F. M. STEWART, Brown University
Ohio, G. M. MERRIMAN, University of Cincinnati
Pacific Northwest, A. T. LONSETH, Oregon State College
Southeastern, G. B. HUFF, University of Georgia
Southwestern, CHARLES WEXLER, Arizona State University
Upper New York State, H. S. M. COXETER, University of Toronto

COMMITTEES OF THE ASSOCIATION

FINANCE COMMITTEE

W. B. CARVER (1956–1959), E. A. CAMERON (1958–1961), H. M. GEHMAN, *ex officio*.

NOMINATING COMMITTEE FOR 1959

MINA REES, *Chairman*, E. G. BEGLE, A. E. TAYLOR.

EDITORIAL COMMITTEE ON CARUS MONOGRAPHS

TIBOR RADO, *Chairman* (1954–1959), E. E. FLOYD (1956–1961), W. R. SCOTT (1957–1959), HARLEY FLANDERS (1958–1960), R. P. DILWORTH (1959–1961).

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COMMITTEE ON EARLE RAYMOND HEDRICK LECTURES

R. P. DILWORTH, *Chairman* (1957–1959), L. H. LOOMIS (1958–1960), A. S. HOUSEHOLDER (1959–1961).

COMMITTEE ON HIGH SCHOOL CONTESTS

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JOINT COMMITTEE ON PLACES OF MEETINGS

R. D. SCHAFER, *Chairman* (1957–1959), G. R. MACLANE (1958–1960), R. H. BRUCK (1959–1961).

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

W. R. SCOTT, *Chairman* (July 1957–1959), RICHARD BELLMAN (1958–1960), IVAN NIVEN (1959–1961), L. E. BUSH, Director (1958–1962).

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COMMITTEE ON SECTIONS

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COMMITTEE ON SLAUGHT MEMORIAL PAPERS

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CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31-September 3, 1959.

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.

ILLINOIS, Millikin University, Decatur, May 8–9, 1959.

INDIANA, Valparaiso University, May 2, 1959.

IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.

KANSAS, Marymount College, Salina, April 11, 1959.

KENTUCKY, Centre College of Kentucky, Danville, April 25, 1959.

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MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Towson, Maryland, May 2, 1959.

METROPOLITAN NEW YORK, Polytechnic Institute of Brooklyn, April 18, 1959.

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MINNESOTA, University of Minnesota, Minneapolis, April 25, 1959.

MISSOURI, Lindenwood College, St. Charles, April 25, 1959.

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NEW JERSEY, Princeton University, November 7, 1959.

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OHIO, Miami University, Oxford, May 9, 1959.

OKLAHOMA, Tulsa University, April 10–11, 1959.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.

PHILADELPHIA, University of Delaware, Newark, November 28, 1959.

ROCKY MOUNTAIN, Utah State University, Logan, May 8–9, 1959.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, Arizona State University, Tempe, April 10–11, 1959.

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UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.

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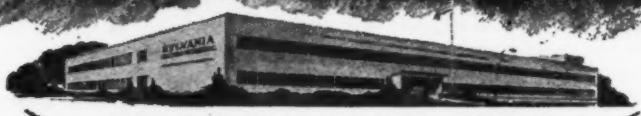

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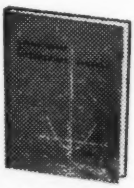

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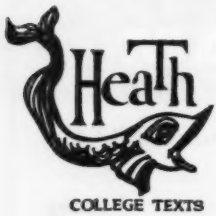

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